On Knaster's Conjecture

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Contents

1 Introduction

1.1 Knaster's Conjecture

A theorem of Borsuk and Ulam says that, for any map of an n-dimensional sphere into Euclidean space of the same dimension, there is a pair of antipodal points whose images coincide, i.e.

Theorem 1 (Borsuk-Ulam) *If $f\colon S^n \to \mathbb{R}^n$ is a continuous map then there exists a pair of antipodal points, $x, -x \in S^n$, such that $f(x) = f(-x)$.*

This theorem was conjectured by S. Ulam and proved by K. Borsuk [1] in 1933. Rephrased the theorem states that for any set of n continuous real-valued functions on the sphere there must be a pair of antipodal points on which all the functions agree. To give a more physical picture, one interpretation for the case $n = 2$ is that there is always a pair of antipodal points in the earth's atmosphere with the same temperature and barometric pressure (assuming, of course, that temperature and pressure vary continuously).
In 1947 Knaster conjectured, as a generalization of this Borsuk-Ulam theorem, the following: Given a mapping from the n-sphere into m-dimensional euclidean space, do there exist $n-m+2$ points on the sphere mapped into a single point which can be rotated to $n-m+2$ given points? Phrased differently:

Conjecture 1 (Knaster) Given a continuous map $f\colon S^n \to \mathbb{R}^m$ and $k = n-m+2$ points $s_1, \ldots, s_k \in S^n$, there exists a rotation $\rho \in SO(n+1)$ such that $f \circ \rho(s_1) = \cdots = f \circ \rho(s_k)$

For more points than the bound $k = n - m + 2$ in the conjecture it is easy to construct counterexamples, see Lemma 23 in Section 6. There are several counterexamples to this conjecture and for some m and k it is proven to be correct.
The conjecture is proved to be correct for the complete case $k = 2$ by Hopf [9] and for the case $k = 3$, $m = 1$, $n = 2$. The cases $m = 1$, $m = 2$ and $k \geq 4$ are still unknown.
The special case $m = 1$, $n = k - 1$ where the $n + 1$ points are placed such that vectors $Os_1, \cdots, Os_{n+1}$, with O the origin of the sphere, are mutally perpendicular is proven to be correct. All other cases of the conjecture are proven to be false by Makeev [13] and Chen [3]. There is an extended literature on this and related problems.

1.2 A survey of the literature on the subject and related problems

Kakutani, S. in [11]. A proof that there exists a circumscribing cube around any bounded closed convex set in $\mathbb{R}^3$. An affirmative answer is given to the following query of Rademacher: Is it possible to circumscribe a cube about a given bounded closed convex set in three-space? The problem is solved by proving that for any real continuous function $f(P)$ defined on a two-sphere there exist points P_1, P_2, P_3, with $f(P_1) = f(P_2) = f(P_3)$ and such that the vectors OP_1, OP_2, OP_3, with O the center of the sphere, are orthogonal.

Hopf, Heinz in [9]. The mapping and covering theorems in question are by:
(1) Borsuk-Ulam, (2) Alexandroff-Hopf and (3) Lusternik-Schnirelmann-Borsuk.
The second theorem asserts: if the n-dimensional sphere S^n is covered by $n + 2$ closed sets

$F_1, F_2, \ldots, F_{n+2}$, of which none contains an antipodal pair of points of S^n, the intersection of any $n+1$ of the F_i is not empty. The author shows that the number π (the angular distance of an antipodal point-pair) has no special force in validating this theorem, so that it remains true when "antipodal pair of points" is replaced by "pair of points at angular distance a," where a is any given number between 0 and π. Furthermore, this theorem generalizes to any closed n-dimensional manifold with a Riemannian metric. Theorems (1) and (3) generalize in the same way, but the main interest in the argument pertains to (2). A discussion of several related unsolved problems is appended.

H. Yamabe and Z. Yujobo in [8]. The following theorem is proved for every dimension n. Let $f(X)$ be a real-valued continuous function defined on an n-dimensional sphere S^n. Then there exist $n+1$ points $X_1, \ldots, X_{n+1}$ on S^n such that
(i) the vectors $OX_1, \cdots, OX_{n+1}$ are perpendicular to one another
and
(ii) $f(X_1) = \cdots = f(X_{n+1})$.
This theorem was originally conjectured by Rademacher and the case $n = 2$ was proved by Kakutani [11]. The authors prove this theorem by reducing it to the following lemma which can be proved by induction on n. Let S_0^n, ${S_1}^n$ be two concentric n-dimensional spheres in $\mathbb{R}^{n+1}$ with the same center O. Let L be a closed subset of $\mathbb{R}^{n+1}$ contained between S_0^n and S_0^n which intersects any continuous curve joining S_0^n and S_1^n. Then L contains $n+1$ points $X_1, \ldots, X_{n+1}$ such that the vectors $OX_1, \ldots, OX_{n+1}$ are perpendicular to one another and have the same length.

Dyson, F. J. in [4]. The author proves that for any real-valued continuous function $f(x)$ defined on a two-sphere S^2 there exist four points x_1, x_2, x_3, x_4 on S^2 forming the vertices of a square on a great circle of S^2 (i.e. x_1, x_2, x_3, x_4 are the end points of two mutually orthogonal diameters of S^2) such that $f(x_1) = f(x_2) = f(x_3) = f(x_4)$. The proof is based on the following lemma: Let L be a closed bounded set in $\mathbb{R}^3$ not containing the origin O of $\mathbb{R}^3$ such that every continuous curve joining O with a point at infinity intersects L. If the intersection D of L and its reflection L' with respect to O is a continuum, then there exist two points x_1 and x_2 in D equidistant from O and sustaining a right angle at O.

Livesay, G. R. in [6]. This note proves the following theorem: Let f be a mapping from the 2-sphere S^2 into a real line. Then for any θ there exist two diameters subtending the angle θ such that the four end-points are mapped to the same point on $\mathbb{R}$.

Yang, C.T. in [20]. The author proves a number of theorems containing ones proved by those whose names are listed in the title. This field started with a conjecture made by Rademacher that there exists a frame on a sphere where a given real-valued function takes the same value. In the paper they try to overcome difficulties in obtaining an extension of Dyson's theorem [4] to the n-sphere by means of a special homology theory. For this purpose the author replaces the sphere by the T-space $(X;T)$, where X is a compact Hausdorff space and T a fixed point free involution on X. The author introduces the (co-)homological index B in explicit form for the space $(X;T)$ by using simplicial approximation. This paper deals with a case where an involutive homeomorphism acts on the sphere. The author establishes a homology theory which is essentially that of the space of orbits of the homeomorphisms.

Among many noteworthy generalizations of classical lemmas, it seems that the most powerful one is Theorem 2, which claims that if f is a map from the space of orbits of an involutive homeomorphism on a finite-dimensional space X into k-dimensional euclidean space $\mathbb{R}^k$, then the dimension of the set of points whose orbit goes to a single point in $\mathbb{R}^k$, is larger than $n-k$. He also proves the following theorem. A continuous real-valued function on an n-sphere maps the $2n$ endpoints of some n mutually orthogonal diameters into a single value.

Floyd, E. in [5]. Floyd proves the case $n = 2$ and $m = 1$ of Knaster's conjecture to be true, i.e. for real valued mappings of the 2-sphere and three points on the sphere. We will give the proof of this in Section 5.

Yang, C.T. in [22]. The author proves theorems concerning mappings from spheres into euclidean spaces. He defines the following: For a T-space $(X;T)$ with a X a compact Hausdorff space, and T a continuous involution on X without fixed point, there is a (unique) non-negative integer n such that a map $f\colon (X;T) \to (S^q;\tilde{T})$, with $\tilde{T}$ the antipodal map on S^q, exist for $q = n$, but not for $q = n-1$. This number is the B-index of $(X;T)$. With the aid of this B-index he proves the following theorem. A map of an $(mn+n-1)$-sphere into euclidean m-space maps the endpoints of some n mutually orthogonal diameters into a single point.

Yang, C.T. in [21]. The author succeeds in proving a theorem which states that for any map from an mn-sphere into m-dimensional euclidean space, there are n diameters on the mn-sphere congruent to n given diameters which are mapped to a single point. This is an extension of theorems the author has proved in his preceding paper, see [20] and [22]. The proof depends upon devices introduced by the author and upon lemmas which explore the relations between point-set topology and combinatorial topology. Connected with this problem is the conjecture by Knaster.

Livesay, G. R in [7]. This note contains the proof of the following theorem: Let $f\colon S^3 \to \mathbb{R}^2$ be a continuous map of the 3-sphere into the Euclidean plane, and let s_1, s_2, and s_3 be points of S^3 which are vertices of an equilateral triangle. Then there exists a rotation ρ of S^3 such that $f(\rho(s_1)) = f(\rho(s_2)) = f(\rho(s_3))$.

Yang, C.T. in [23]. The author extend the result of Kakutani [11], by removing the restriction on n and weakening the restriction on x_1, x_2, x_3, by proving the following lemma. For all continuous maps f between S^{n+1} in $\mathbb{R}^n$ and for any number d, $0 < d \leq 2\pi/3$, there are three points x_1, x_2, $x_3 \in S^{n+1}$, such that $f(x_1) = f(x_2) = f(x_3)$ and the spherical distance between every two of the three points x_1, x_2, x_3 is equal to d.

Makeev, V. V. in [13] (Russian), translation in [14]. The author reviews the results to Knaster's conjecture obtained in various special cases. He also gives an affirmative answer in the case when the points $s_1, \ldots, s_k$ are vertices of a regular polygon with $k+1$ vertices inscribed into a great circle of the sphere S^{k-1} (for $k \geq 4$ and $k+1$ a prime). The proof relies on some results concerning equivariant maps. The rest of the paper deals with the set $\{\rho \in SO(n) | f(\rho(s_1)) = \cdots = f(\rho(s_k))\}$, with $f\colon S^{n-1} \to \mathbb{R}^m$. The author offers a short review of the known results and adds a few new results; he also lists some open problems.

Makeev, V. V. in [16] (Russian), translation in [17]. The author improves on his earlier results for $m = 1$ by proving that a configuration $\{s_1, \ldots, s_k\}$ is a solution to Knaster's problem if either of the following sufficient conditions holds:
(a) k is a prime, $k \leq 2n$, and $s_1, \ldots, s_k$ are the vertices of a regular k-gon inscribed in a great circle of S^n,
(b) $k+1$ is a prime, $k = 2n$, and $s_1, \ldots, s_k$ are vertices of a regular $(k+1)$-gon inscribed in a great circle of S^{n-1}.
A weaker result is obtained for $m = 2$:
if $s_1, s_2, s_3 \in S^2$ are the vertices of a regular triangle, then for each $f, g \colon S^2 \to \mathbb{R}$ there exists a rotation ρ of S^2 such that $f(\rho(s_1)) = f(\rho(s_2)) = f(\rho(s_3))$ and $g(\rho(s_1)) = g(\rho(s_2))$.
The proofs involve the derivation of certain properties of real-valued maps of the Stiefel manifolds $V_{2,n}$. The author also formulates an infinitesimal version of Knaster's problem.

Remark 1 Some of the results in this paper have also been derived in another of the author's papers [13]. The set of k-frames in $\mathbb{R}^n$ is a manifold (see for explanation Section 3) and is called the **Stiefel manifold** $V_{n+1,l}$. The dimension of the space which is spanned by the vectors $Os_1, \ldots, Os_k$, with O the center of the sphere, will be denoted by l and we have $l \leq \min\{k, n+1\}$. Makeev proves in [13] and [15], the following general result:

Theorem 2 *If dim $V_{n+1,l} < m(k-l)$ then the conclusion of Knaster's conjecture is false.*

Makeev conjectured that the inequality dim $V_{n+1,l} \geq m(k-l)$ is sufficient for the conclusion of Knaster's conjecture to be true. This inequality is called Makeev's inequality.

Babenko, I. Bogatyĭ, S. A. in [10]. The paper shows how to construct counterexamples to Knaster's conjecture.
They look at the $(l-1)$-dimensional sphere S^{l-1}, linearly embedded in $\mathbb{R}^n$ and at the space of maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ defined by homogenuous polynomials of a certain degree t, $\mathcal{P}_{n,t}$. They show that in the space $\mathcal{P}_{n,t}$, for some conditions of n,m and k depending on the degree t and the dimension of this space, the set of maps not constant on an l-dimensional k-point configuration is open and dense. They state that from this one can construct a counterexample to Knaster's conjecture and that satisfies Makeev's inequality, so it is also a counterexample to Makeev's conjecture.

Chen, William in [3]. The author gives several counterexamples to Knaster's conjecture, thus substantially extending the list of previously known counterexamples found by V. V. Makeev [13], and by I. K. Babenko and S. A. Bogatyi [10]. The examples constructed by Chen also provide counterexamples to the Makeev conjecture. The main theorem of the paper is the following: Knaster's conjecture is false for all $n > m > 2$. In particular, a special case of the Knaster conjecture attributed to Spanier, see [12], is answered by the above theorem. At the end the author includes a very informative table listing the known affirmative results and counterexamples related to Knaster's conjecture and states the following conjecture (Chen): Knaster's conjecture is false for $n > m > 1$.

Volovikov, A. Yu. in [18]. Volovikov proofs Knaster's conjecture in the affirmative for $m = 1$ and $m = 2$ in case of special configurations of points, namely Z_p-symmetric sets of p^n or $p(n+1)$ points for $m = 1$ or Z_p-symmetric sets of p^n points for $m = 2$, where $p > 2$ is a prime number. With Z_p-symmetric points they mean that this set of points is invariant under a certain Z_p-action on the sphere.

Volovikov, A. Yu. in [19]. In that paper, Knaster's conjecture is proved for the case in which $n = p^2$ for an odd prime p and the points lie on a circle and divide it into equal parts.

1.3 The work in this thesis

William Chen gives in [3] six counterexamples to Knaster's conjecture. The six examples give a general result:

Theorem 3 (Chen) *Knaster's conjecture is false for all $n > m > 2$.*

We will discuss this result of Chen in Section 4. The cases $m = 1$, $m = 2$ and $k \geq 4$ are still unknown. The case $m = n$, $k = 2$ was answered in the affirmative by Hopf in [9] as a generalization of the Borsuk-Ulam theorem. The case $m = 1$, $k = 3$ was proved by E.E. Floyd in [5] by set theoretical arguments.
These two cases are the closest to the unknown cases $k = 4$ and $m = 1$, $m = 2$. Therefore we will discuss both papers. We will make the proof of several generalizations done by Hopf and Floyd more accessible. Futhermore we try to use it in order to prove or disprove the unknown cases. In Sections 2 and 3 we will give the reader a short introduction to the mathematical background in order to understand those proofs.

2 Homology

2.1 Introduction to homology

The fundamental group $\pi_1(X)$ of a topological space X is especially useful when studying spaces of low dimension. The higher-dimensional homotopy groups have the serious drawback that they are extremely difficult to compute in general. Fortunately there is a more computable alternative to homotopy groups: the homology groups of X, denoted by $H_n(X)$. The computability of homology groups does not come for free, unfortunately. The definition of homology groups is definitely less transparent than the definition of homotopy groups, and even when one gets beyond the definition there is a certain amount of technical machinery to be set up before any real calculations and applications can be given. The first homology theory we describe here is based on spaces which admit a so-called triangulation. These are spaces which can be built from glueing together simplices by certain rules. We will define the simplex in the next subsection. In this way one obtains a homology theory, called simplicial homology built from simplices. Still, this is a rather restricted class of spaces, and the theory itself has a certain rigidity that makes it awkward to work with. The way to overcome these obstacles is to step back from the geometry of spaces decomposed into simplices and to consider instead something which at first glance seems more complicated, the collection of all possible continuous maps of simplices into a given space X. With these maps we obtain the so-called singular homology groups.

2.2 Simplices

We begin with an example: the projective plane. This surface can be obtained from a square by identifying opposite edges, in the way indicated by the arrows on the boundary of the square. If we cut the square along a diagonal, we get two triangles, so the projective plane can also be constructed from two triangles by identifying certain pairs of edges. This is not a triangulation though, therefore we need a more complicated construction. Using only triangles we could also construct a large class of 2-dimensional spaces that are not surfaces in the strict sense, by allowing more than two edges to be identified together at a time. Δ^n-complexes are a generalization of this idea, using the n-dimensional analogue of the triangle, the (affine) n-simplex. This is the smallest convex set in $\mathbb{R}^m$ with $m > n$ containing $n+1$ points $v_0, \dots, v_n$ that do not lie in a hyperplane of dimension less than n.
An equivalent condition would be that the vectors $v_1 - v_0, \dots, v_n - v_0$ are linearly independent. The points v_i are the vertices of the simplex, and the simplex itself will be denoted $[v_0, \dots, v_n]$. This leads to the definition of the so-called standard n-simplex.

Definition 1 (standard n-simplex) The **standard n-simplex** in $\mathbb{R}^m$ with $m > n$:

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^m \,|\, \sum_{i=0}^{n} t_i = 1, \quad t_i \geq 0 \quad \forall i\}$$

The vertices of Δ^n are the unit vectors along the coordinate axes.

The coefficients t_i are called the **barycentric** coordinates. A **face** of a simplex $[v_0, \dots, v_n]$ is a subsimplex with any nonempty subset of the v_i's as its vertices.

To build topological spaces out of n-simplices will be very hard. In order to make it more flexible, i.e. to define homology theory for more topological spaces, we will work with the so-called affine singular n-simplex. It is almost the same definition of the n-simplex described above, but we drop the restriction that the v_i are assumed to be independent, so it can be degenerate.

Definition 2 (affine singular n-simplex) Given points $v_0, \dots, v_n \in \mathbb{R}^m$ with $m > n$, let $[v_0, \dots, v_n]$ denote the map $\Delta^n \to \mathbb{R}^n$ defined by $t_0 e_0 + \dots + t_n e_n \mapsto t_0 v_0 + \dots + t_n v_n$.

In order to glue these simplices together we need to define the boundary of a simplex, which is built out of simplices of one dimension less. We define the face of an affine singular n-simplex as the map:

Definition 3 (i^{th} face map) The simplex $[v_0, \dots, \hat{v_i}, \dots, v_n] : \Delta_{n-1} \to \Delta_n$ is called the i^{th} **face** map and is denoted by F_i^n. The hat on the i^{th} vector v_i, means that that vector is omitted.

The subset need not be a proper subset; $[v_0, \dots, v_n]$ is regarded as a face of itself. We adopt the convention that the vertices of a face will always be ordered according to their order in the larger simplex.

2.3 Singular Homology

The most important homology theory in algebraic topology is called singular homology, which we will introduce here. It is based on simplicial homology, in which the 'physical' triangle is substituted by continuous maps of a standard triangle.

Definition 4 If X is a topological space then a **singular n-simplex** of X is a map $\sigma_n \colon \Delta^n \to X$. The **singular n-chain group** $\Delta_n(X)$ is the free abelian group based on the singular n-simplices.

Free abelian group based on the singular n-simplices, is a group based on formal finite sums, $c = \sum_\sigma n_\sigma \sigma$, with n-simplices as a basis. A finite sum c of n-simplices, is called an n-chain.

Definition 5 If X is a topological space then the i^{th} face of σ is $\sigma^{(i)} = \sigma F_i^n$. The boundary of σ is $\partial_n \sigma = \sum_{i=0}^n (-1)\sigma^{(i)}$, an $(n-1)$-chain. If $c = \sum_\sigma n_\sigma \sigma$ is an n-chain, then we define $\partial_n c = \partial_n(\sum_\sigma n_\sigma) = \sum_\sigma n_\sigma \partial_n \sigma$.

In order to define the homology groups, we need the following lemma and definitions.

Lemma 4 *The composition $\partial_n \partial_{n+1} = 0$.*

From this lemma we have that $\partial_{n+1}(\Delta_{n+1}(X)) \subset \text{kernel}(\partial_n \Delta_n(X))$.

Definition 6 The chains of Δ_n which lie in the image of ∂_{n+1} are called **n-boundaries**, or just **boundaries**. We denote the subset Δ_n of n-boundaries as $B_n(X)$. The chains of Δ_n which lie in the kernel of ∂_n are called **n-cycles**, or just **cycles**. We denote the subset containing n-cycles of Δ_n as $Z_n(X)$.

By Lemma 4, we have $B_n(X) \subset Z_n(X)$. Then the following quotient makes sense:

Definition 7 (singular homology group) The p^{th} **singular homology group** of a space X is:

$$H_n(X) = Z_n(X)/B_n(X).$$

The equivalence class of a cycle c is denoted $[c] \in H_n(X)$. Two cycles $c_1, c_2 \in [c]$ are called homologous. For $c_1, c_2 \in [c]$ we have $c_1 - c_2 = \partial d$ where d is some $n+1$ chain, i.e. they differ a boundary. Homology groups are invariant under homeomorphisms, i.e. if two topological spaces X, Y are homeomorphic then their homology groups $H_n(X), H_n(Y)$ are isomorphic.

2.4 Maps between homology groups

Suppose we have a map $f \colon X \to Y$ between the topological spaces X and Y, then we want to define a homomorphism between both their homology groups $H_n(X), H_n(Y)$, called the induced homomorphism (or map).
For a continuous map $f \colon X \to Y$, we can take for any singular n-simplex $\sigma \colon \Delta^n \to X$

the composition $f \circ \sigma \colon \Delta^n \to Y$ and obtain a singular n-simplex into Y. For a chain $c = \sum_\sigma n_\sigma \sigma \in \Delta^n$, we turn this map f into a homomorphism f_Δ by setting:

$$f_\Delta(\sum_\sigma n_\sigma \sigma) = \sum_\sigma n_\sigma (f \circ \sigma) \tag{1}$$

Lemma 5 *For a map $f \colon X \to Y$, the induced homomorphism $f \colon \Delta^n(X) \to \Delta^n(Y)$ commutes with the boundary map, i.e. $f_\Delta \circ \partial = \partial \circ f_\Delta$*

We need this lemma in order to have the definition of the induced homomorphism on the level of homology groups to be well-defined. A homomorphism f is well-defined if the map does not depend on the choice of a representant of an equivalence class. This means that for $c, \tilde{c} \in [c]$ we have that $f(c - \tilde{c}) \in [0]$. Because two cycles are homologous if they differ a boundary we have, by Lemma 5, that the following definition is well-defined:

Definition 8 A map $f \colon X \to Y$ induces the homomorphism $f_* \colon H_n(X) \to H_n(Y)$ defined by $f_*([c]) = [f_\Delta(c)]$.

The relations,

$$(f \circ g)_* = f_* \circ g_*$$
$$id_* = id$$

are immediately from the definition.

2.5 The degree of a map from the n-sphere into the n-sphere

The n^{th} singular homology group of the n-sphere, $H_n(S^n)$, is an infinite cyclic group, hence there are two different choices possible for a generator of this group.

Definition 9 If a choice of a generator has been made for $H_n(S^n)$, then S^n is called oriented.

The knowledge about the homology groups of S^n will be used to study continuous maps of S^n into itself. Let $f \colon S^n \to S^n$ be such a continuous map and

$$f_* \colon H_n(S^n) \to H_n(S^n)$$

be the induced homology map. Since $H_n(S^n)$ is an infinite cyclic group, there exists a unique $d \in \mathbb{Z}$ such that $f_*(u) = du \quad \forall u \in H_n(S^n)$.

Definition 10 (degree) This integer d is called the **degree** of $f_* \colon H_n(S^n) \to H_n(S^n)$.

Some basic properties of the degree d of a map $f_* \colon H_n(S^n) \to H_n(S^n)$ are:

1. The degree has the property that it is homotopy invariant. This means that the degree of f will not change under continuous deformations of f.

2. If the degree of f is nonzero then f is surjective. The reason for this is that S^n minus one point is homotopic to $\mathbb{R}^n$. If f is not surjective we can factor f through a map from $H_n(S^n)$ into $H_n(\mathbb{R}^n) = 0$.

3. From properties 1 and 2 it follows that a map with nonzero degree is not homotopic to a map with zero degree.

4. Let f be a reflection, i.e. a map which is the reflection in a hyperplane through the origin of $\mathbb{R}^{n+1}$. Reflection means the change of orientation of S^n. This means a change of the choice of one of the generators of $H_n(S^n)$. Thus $d = -1$.

5. If f is the antipodal map, then the $d = (-1)^{n+1}$. The antipodal map is is the composition of $n+1$ reflections $\{r_i\}_{i=1}^{n+1}$ with

$$r_i(x_1, \ldots, x_i, \ldots, x_{n+1}) = r_i(x_1, \ldots, -x_i, \ldots, x_{n+1}).$$

3 Topological Groups

In this chapter we will introduce the notion of a topological group, which forms a rich source of interesting spaces. Topological groups are topological spaces with an additional group structure.

Definition 11 (Topological group) A topological group is a Hausdorff topological space G together with a group structure compatible in the sense that:

- group multiplication $(g, h) \to gh$ on G is continuous
- group inversion $g \mapsto g^{-1}$ on G is continuous.

3.1 Examples of topological groups

Topological groups form a particularly nice family of spaces, since it turns out that the group structure imposes severe restrictions on the topology. There are many natural examples of topological groups. We will mention a few here.
Examples of topological groups:

- The additive group of the real numbers $\mathbb{R}^+$ is a connected topological group.
- The multiplicative group $\mathbb{R}^* = \mathbb{R} - 0$, forms a non-connected, in the topological sense, topological group.
- The circle, S^1, thought of as the set of complex numbers of unit modulus. The circle is then a subgroup of the multiplicative group of the complex numbers $\mathbb{C}^*$. The subgroup (as well as the subspace) given by $S^1 = z \in \mathbb{C} : |z| = 1$. Notice that S^1 is a subgroup by virtue of the identities $|z_1 z_2| = |z_1||z_2|$ and $|z^{-1}| = |z|^{-1}$. The topology is the subspace topology and the group structure is multiplication of complex numbers, i.e. the functions:

$$S^1 \times S^1 \to S^1, \ (e^{i\theta}, e^{i\phi}) \mapsto e^{i(\theta+\phi)}, \tag{2}$$
$$S^1 \to S^1, \ (e^{i\theta}, e^{i\phi}) \mapsto e^{-i\theta}, \tag{3}$$

are continuous.

- The 3-sphere, S^3, considered as the unit sphere in the space of quaternions $\mathbb{H}$.
- The group of invertible $(n \times n)$-matrices with real entries. The group structure is matrix multiplication. For the topology we identify each $(n \times n)$-matrix $A = (a_{ij})$ with the corresponding point $(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, a_{31}, \ldots, a_{nn})$ of $\mathbb{R}^{n^2}$ and take the subspace topology. Multiplication of matrices is a polynomial in the entries of both matrices. Taking the inverse of a matrix is also a polynomial in the entries of the matrix, so both group multiplication and group inversion are continuous in this topology.
  This topological group is called the general linear group, and is denoted by $GL(n)$. There are many interesting subgroups of $GL(n)$, which are, with the induced topology, also topological groups. An example of these groups is the group of orthogonal matrices $O(n)$, which we will define and discuss in detail in the next Subsection.

3.2 The orthogonal group

The subgroup $O(n)$ of $GL(n)$ is called the orthogonal group and consists of all linear transformations which take any orthonormal basis for $\mathbb{R}^n$ to another orthonormal basis. An equivalent definition is the group of isometries of $\mathbb{R}^n$ fixing the origin. We will use a third equivalent definition of the orthogonal group $O(n)$:

Definition 12 (orthogonal group) $O(n) = \{A \in GL(n) : A^t A = 1\}$.

From this it follows that $O(n)$ is the subgroup of all matrices whose columns form an orthonormal basis for $\mathbb{R}^n$. We can construct the subgroup of $O(n)$, by taking all matrices of determinant 1, so we obtain the special orthogonal group:

Definition 13 (special orthogonal group) $SO(n) = \{A \in O(n) : det(A) = 1\}$.

The following theorem we will need it in the next subsection in order to define the Stiefel manifold.

Theorem 6 *$O(n)$ and $SO(n)$ are compact.*

PROOF We will show that $O(n)$ is compact, by showing that it is a closed, bounded subset of $\mathbb{R}^{n^2}$ under the identification of $GL(n)$ with $\mathbb{R}^{n^2}$. Let $A = (a_{ij}) \in O(n)$. Since $A^t A = 1$, we have the following equations:

$$\sum_{j=1}^{n} a_{ij} a_{kj} = \delta_{ik} \quad 1 \leq i, k \leq n.$$

For each choice of i, k we define the map f_{ik} from the space of all matrices to $\mathbb{R}$ by $f_{ik}(A) = \sum_{j=1}^{n} a_{ij} a_{kj}$. Then $O(n)$ is the intersection of all the closed sets of the form:

$$\begin{array}{ll} f_{ik}^{-1}(0) & 1 \leq i, k \leq n, \quad i \neq k \\ f_{ii}^{-1}(1) & 1 \leq i \leq n \end{array}$$

and therefore closed. The equation (4) implies that the entries of any orthogonal matrix satisfy $|a_{ij}| \leq 1$. Thus $O(n)$ is a bounded, closed subset of $\mathbb{R}^{n^2}$ and thus compact. $SO(n)$ is compact, because it is closed in $O(n)$. $\square$

Since the columns of a matrix in $O(n)$ are unit vectors, $O(n)$ can also be regarded as a subspace of the product of n copies of S^{n-1}. Because an orthogonal matrix, as a transformation of Euclidean n-space, is an isometry, it is also a map from S^{n-1} to itself. We can regard $O(n-1)$ as the subgroup of $O(n)$ fixing the last coordinate. Then such matrix of $O(n-1) \subset O(n)$ is of the form:

$$B = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ 0 & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

The vector $e_1 = (1, 0, \dots, 0)^t$ is mapped on itself by any matrix of $O(n-1)$. We define the map:

$$f \colon O(n) \to S^{n-1} \text{ by } f(A) = Ae_1$$

Let $p \colon O(n) \to O(n)/O(n-1)$ be the projection map. If $B \in O(n-1)$, then $f(AB) = f(A)$. This means that we can define a map $g \colon O(n)/O(n-1) \to S^{n-1}$, such that $f = g \circ p$. This induced map $g \colon O(n)/O(n-1) \to S^{n-1}$ is bijective and continuous and maps a compact space into a Hausdorff space and is therefore a homeomorphism. We shall generalize this in the next subsection.

3.3 Orbit spaces

Definition 14 If G is a topological group and X is a topological space, than an **action** of G on X is a map $G \times X \to X$, with the image of (g, x) denoted by $g(x)$, such that:

- $(gh)(x) = g(h(x))$
- $e(x) = x$

With the notion of action of a topological group on a topological space come several other definitions. For a fixed point x of X we define:

Definition 15 The **orbit** of $x : G(x) = \{g(x) | g \in G\} \subset X$.

Definition 16 The action is **transitive** if there is only one orbit, the whole space X.

Definition 17 The **isotropy** or **stability group**: $G_x = \{g \in G | g(x) = x\} \subset G$.

Now we can give a theorem, without proof, which generalizes our comments on $O(n)$ acting on S^{n-1}.

Theorem 7 *If G is a compact topological group acting on the Hausdorff space X and G_x is the isotropy group at x, then the map $\phi\colon G/G_x \to G(x)$ given by $[g] \to g(x)$ is a homeomorphism.*

We use this theorem to give a homeomorphism of the so called Stiefel manifold. First we shall give a definition of a k-frame, which we use to construct the Stiefel manifold. From here on we take $k \leq n$.

Definition 18 (k-frame) A **k-frame** is a set of orthonormal vectors in $\mathbb{R}^n$-space. A k-frame can then be regarded as an element of the k-fold cartesian product of S^{n-1}, $S^{n-1} \times \cdots \times S^{n-1}$.

Definition 19 The **Stiefel manifold** $V_{n,k}$ is the set of all k-frames in $\mathbb{R}^n$. The Stiefel manifold is then a subset of k copies of S^{n-1}, $S^{n-1} \times \cdots \times S^{n-1}$, and is equipped with the subspace topology.

The columns of an orthogonal matrix are pairwise orthogonal vectors and so they form an n-frame. Thus a k-frame can be seen as an element of $O(k)$. We can extend a k-frame with $n-k$ basis vectors in order to get an n-frame. In this way we can also regard a k-frame as an element of $O(n)$. $O(n)$ acts transitively on $V_{n,k}$. The stability group is $O(n-k)$. Then we have by Theorem 7 that $O(n)/O(n-k)$ is homeomorphic to $V_{n,k}$. The same holds for $SO(n)$, i.e. $SO(n)/SO(n-k)$ is homeomorphic to $V_{n,k}$.

3.4 The special orthogonal group

In this subsection we will give some insight in the group of rotations $SO(n)$, by describing (by homeomorphisms) $SO(n)$ up to $n = 4$ in terms of more familiar spaces.

- $SO(1)$ is the trivial group.
- $SO(2)$, the group of rotations of $\mathbb{R}^2$, is both homeomorphic and isomorphic as a group to S^1, thought of as the unit complex numbers.
- $SO(3)$ is homeomorphic to $\mathbb{R}P^3$. To see this, let $f\colon D^3 \to SO(3)$ send a nonzero vector x to the rotation through angle $|x|\pi$ about the axis formed by the line through the origin in the direction of x. An orientation convention such as the right-hand rule is needed to make this unambiguous. By continuity, f then sends zero to the identity. Antipodal points of $S^2 = \partial D^3$ are sent to the same rotation through angle π, so f induces a map $\overline{f}\colon \mathbb{R}P^3 \to SO(3)$, regarding $\mathbb{R}P^3$ as D^3 with antipodal boundary points identified. The map $\overline{f}$ is clearly injective, since the axis of a nontrivial rotation is uniquely determined as its fixed point set, and $\overline{f}$ is surjective since by linear algebra each nonidentity element of $SO(3)$ is a rotation about some axis. It follows that $\overline{f}$ is a homeomorphism, since $\mathbb{R}P^3$ is compact and $SO(3)$ is Hausdorff.
- $SO(4)$ is homeomorphic to $S^3 \times SO(3)$. Identifying $\mathbb{R}^4$ with the quaternions $\mathbb{H}$ and S^3 with the group of unit quaternions, the quaternion multiplication $v \mapsto vw$ for fixed $w \in S^3$ defines an isometry $\phi_w \in SO(4)$, since $|vw| = |v||w| = |v|$ if $|w| = 1$.

Points of $O(4)$ are 4 tuples $(v_1, \cdots, v_4)$ of orthonormal vectors $v_i \in \mathbb{H} = R^4$, and we view $SO(3)$ as the subspace with $v_1 = 1$. A homeomorphism $S^3 \times SO(3) \to O(4)$ is defined by sending $(v, (1, v_2, v_3, v_4))$ to $(v, v_2 v, v_3 v, v_4 v) = \phi_v(1, v_2, v_3, v_4)$, with inverse $(v_1, v_2, v_3, v_4) \mapsto (v, (1, v_2 v^{-1}, v_3^{-1}, v_4^{-1})) = (v, \phi_{v^{-1}}(v, v_2, v_3, v_4)$. This is not a group isomorphism, however.

We will use the fact that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$ in Theorem 21.

4 Generalizations of the Borsuk-Ulam theorem

Hopf makes generalizations of the following theorems.

Theorem 8 (Borsuk-Ulam) *If $f\colon S^n \to \mathbb{R}^n$ is a continuous map then there exists a pair of antipodal points, i.e $x, -x \in S^n$ such that $f(x) = f(-x)$*

Theorem 9 *If S^n is covered with $n+2$ closed sets $\{F_i\}_{i=1}^{n+2}$, such that none of these sets contains a pair of antipodal points, then every combination of $n+1$ sets of the covering has a nonempty intersection.*

From this theorem the following theorem follows easily:

Theorem 10 *If S^n is covered with $n+1$ closed sets then at least one of those sets contains a pair of antipodal points.*

Proof Add the empty set to the covering. Then the conclusion of Theorem 9 does not hold and thus also the hypothesis does not hold. $\square$

4.1 The first generalization of the Bosuk-Ulam theorem

Let M be a Riemannian manifold with complete metric and let $b \in \mathbb{R}$ be a fixed, arbitrary positive number. We note the fact that a curve on M is a geodesic if and only if it satisfies a certain system of differential equations in every coordinate neighborhood. A consequence of the fact that the geodesics are characterized by such a system of differential equations is the following lemma.

Lemma 11 *Given an arbitrary point $p \in M$ and a vector $w \in T_p(M), w \neq 0$, then there exist an $\epsilon > 0$ and a unique parametrized geodesic $\gamma\colon (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p, \dot{\gamma}(0) = w$. This curve can be extended forever if the metric is complete and so it can have any length.*

This lemma will be used in the proof of Theorem 13.

Definition 20 An **antipodal preserving map** $f\colon S^n \to S^n$ is a map with $f(-x) = -f(x)$.

For the proof of Theorem 13 we will also need the following fact:

Proposition 12 *Every antipodal preserving map, cannot be deformed continuously into a map from S^n to S^n which is not surjective.*

Proof The degree of a continuous antipodal preserving map f is nonzero, see Bredon [2]. A continuous map $g\colon S^n \to S^n$ which is not surjective has degree zero. Continuous deformation of a map preserves the degree of that map, and so f cannot be deformed into g. □

Theorem 13 *Let M^n be a closed Riemannian manifold and let $f\colon M^n \to \mathbb{R}^n$ be a continuous map. Then for any $b \in \mathbb{R}$ with $b > 0$, there exists in M^n a geodesic curve of lenght b whose endpoints are mapped by f to the same image point.*

Proof Let (E, μ, M^n) be the unit sphere bundle over M^n with projection map μ. The fiber above a fixed point p of M^n is then an $(n-1)$-dimensional sphere, which we denote by $S(p) \cong \{p\} \times S(p)$. We define the antipodal map:

$$\begin{aligned} J\colon E &\to E \qquad\qquad\qquad\qquad r \in S(p) \\ (p,r) &\mapsto (p,-r) \end{aligned} \tag{4}$$

Each $r \in S(p)$ defines a geodesic curve in M^n, i.e (see Lemma 11)
for all $r \in S(p)$ there exists a geodesic curve $\gamma_r\colon \mathbb{R} \to M^n$ such that:

$$\begin{aligned} \gamma_r(0) &= p, \\ \dot{\gamma}_r(0) &= r. \end{aligned}$$

This geodesic curve can have lenght b, because M^n is closed and therefore complete. Define the map (geodesic flow):

$$\begin{aligned} T_s\colon E &\to E \\ (p,r) &\mapsto (\gamma_r(s), \dot{\gamma}_r(s)) \end{aligned}$$

T_s is continuous with respect to r and parameter s.

Remark 2 The curve γ_r is equal to the curve $\gamma(-r)$, but with reversed parametrization, i.e. $\gamma_r(s) = \gamma_{-r}(-s)$ and $\dot{\gamma}_r(s) = -\dot{\gamma}_{-r}(-s)$.

By the Remark we have:

$$T_s J(p,r) = T_s(p,-r) = (\gamma_{-r}(s), \dot{\gamma}_{-r}(s)) = (\gamma_r(-s), -\dot{\gamma}_r(-s)) = JT_{-s}(p,r)$$

and get the following equation between the maps T_s and the antipodal map J:

$$T_s J = JT_{-s}. \tag{5}$$

For the fixed lenght b, we define the continuous map: $F = f \circ \mu\colon E \to \mathbb{R}^n$, and the points: $A_s(p,r) = FT_s(p,r)$, $B_s(p,r) = FT_{b+s}(p,r)$ and denote the difference vector: $v_s(p,r) = B_s(p,r) - A_s(p,r)$. Then we have the following:

1. $v_s(p,r)$ is continuous with respect to r and s.
2. $A_s(p,r), B_s(p,r) \in F(M^n)$ and $A_0 = f(p)$.
3. $v_s(J(p,r)) = -v_s(p,r)$, when $s = -\frac{1}{2}b$. This is because reversed parametrization reverses the endpoints of the geodesic curve γ_r.
4. $v_s(p,r) = 0$ if and only if $f(\gamma_r(s+b) = f(\gamma_r(s))$.

Proof of (4):

$$\begin{aligned}
0 &= v_s(p,r) \\
&= B_s(p,r) - A_s(p,r) \\
&= FT_s(p,r) - FT_s(p,r) \\
&= f \circ \mu(T_{s+b}(p,r)) - f \circ \mu(T_s(p,r)) \\
&= f \circ \mu((\gamma_r(s+b), \dot{\gamma}_r(s+b)) - f \circ \mu((\gamma_r(s), \dot{\gamma}_r(s)) \\
&= f(\gamma_r(s+b)) - f(\gamma_r(s))
\end{aligned}$$

Assume that a geodesic segment as postulated by the theorem does not exist. Then $v_s(p,r) \neq 0$ for all $(p,r) \in E$ and for all $s \in \mathbb{R}$. Then for all $s \in \mathbb{R}$ the following maps are well-defined:

$$V_s \colon E \to S^{n-1}, \quad (p,r) \mapsto \tfrac{v_r(s)}{|v_r(s)|} \tag{6}$$

The maps V_s are continuous with respect to s and r by (1). From (3) it follows that $V_{-\frac{b}{2}}|_{S(p)} \colon S(p) \to S^{n-1}$ is an antipodal preserving map.
We write $f = (f_1, \ldots, f_n)$ with coordinate functions $f_i \colon M^n \to \mathbb{R}$. Because M^n is closed, the coordinate function f_n has a maximum in M^n. We denote $p \in M^n$ as this maximum. We have $v_0(p,r) = B_0(p,r) - f(p)$. Then the n^{th} component of v_0 is not positive. This means that $V_0|_{S(p)}(S(p))$ lies only on one half of S^{n-1}, in particular, $V_0|_{S(p)} \colon S(p) \to S^{n-1}$ is contractible. But then $V_{-\frac{b}{2}}|_{S(p)}$, which is an antipodal preserving map, is continuously deformed by parameter s into the map V_0, which is in contradiction to Proposition 12.
□

Lemma 14 *Let $f \colon S^n \to \mathbb{R}^m$ be a continuous map then for $k = 2$ points s_1, s_2 the conclusion of Knaster's conjecture is true.*

Proof Let $M^n = S^n$ and $b = d_{S^n}(s_1, s_2)$. Then by Theorem of Hopf (Theorem 13)) there are two points x_1, x_2 with $d_{S^n}(x_1, x_2) = b$ and $f(x_1) = f(x_2)$. We then can rotate S^n by a rotation ρ such that $x_i = \rho(s_i)$, $i = 1, 2$. □

Lemma 15 (Urysohn Lemma) *Let X be a normal space; let A and B be disjoint closed subsets of X. Let $[a,b]$ be closed interval in the real line. Then there exists a continuous map*

$$f \colon X \to [a,b]$$

such that $f(x) = a$ for every $x \in A$, and $f(x) = b$ for every $x \in B$.

Theorem 16 *Let M^n be covered by $n+2$ closed sets, $F_1, \dots, F_{n+2}$. None of these sets contains the endpoints of a geodesics curve of length b. Then every $n+1$ of the sets F_i have a non-empty section.*

PROOF M^n is covered with $n+2$ closed sets $F_1, \dots, F_{n+2}$ such that the intersection of all the $n+2$ sets f_i is empty. For if there is at least one point, say x, in this intersection, then for any fixed length b we can find an endpoint y and a geodesic curve of length b between it. But then $x, y \in F_i$ for some i, which is in contradiction of the assumptions of our theorem.

M^n is normal space, because it is a metric space and then we can use the Urysohn Lemma (Lemma 15) for the closed sets $A = F_i$ with $i \in \{1, \dots, n+2\}$ and $B = F_1 \cup \dots \cup F_{i-1} \cup F_{i+1} \cup \dots \cup F_{n+2}$. Then for all $i \in \{1, \dots, n+2\}$ we can define a continuous function

$$f_i \colon M^n \to \mathbb{R}$$

with

$$\begin{aligned} f_i|_{F_i} &\equiv 0 \\ f_i &> 0 \quad \text{otherwise} \end{aligned}$$

and such that

$$\sum_{i=1}^{n+2} f_i(x) = 1.$$

Let T_{n+1} be a simplex with corners $e_1 \dots e_{n+2}$. Let t_i^n be the boundary simplex of T_{n+1} such that $e_i \notin t_i^n$. Notice that the boundary ∂T_{n+1} is homeomorphic to S^n.
By the definition of the f_i, the map:

$$f \colon M^n \to T_{n+1}, \quad f(x) = (f_1(x), \dots, f_{n+2}(x))$$

is well defined. The map f is like a weight function with f_i the weights at e_i, or

$$f(x) = \sum_{i=1}^{n+2} f_i(x)\, e_i.$$

Then we have the following:

$$f(x) \in \partial T_{n+1} \tag{7}$$

$$f(x) \in t_i^n \text{ if and only if } f_i(x) = 0 \text{ if and only if } x \in F_i. \tag{8}$$

From (7) and (8) it follows that:

Remark 3

$$f^{-1}(e_i) = F_1 \cap \dots \cap F_{i-1} \cap F_{i+1} \cap \dots \cap F_{n+2} \tag{9}$$

and from $f(x) = f(x')$ it follows that there exists an F_i with $x, x' \in F_i$.

If $f(M^n) \subsetneq S^n$ then one can construct a map from M^n into $\mathbb{R}^n$. Then by Theorem 13 there exists a geodesic curve with length b, whose endpoints are mapped by f to the same image point. From Remark 3 those two endpoints are in the same closed subset of the covering, which is in contradiction with the assumptions of this theorem.
Hence the sphere S^n is completely covered by $f(M^n)$, when for all $i \in \{1, \dots, n+2\}$ there is a point x_i with $f(x_i) = e_i$. From equation (9) the conclusion of the theorem follows.
□

The analogue of Theorem 10 is then formulated as follows:

Theorem 17 *If M^n is covered with $n+1$ closed sets then at least one of those sets contains the endpoints of a geodesic curve with length b.*

The proof is analoguous to the proof of Theorem 10.

4.2 The second generalization of the Borsuk-Ulam theorem

In Theorem 13 we only know that one geodesic satisfying Theorem 13 exists but not how many. Hopf defines $\mathcal{B}$ as a subset of E such that for all $(p,r) \in \mathcal{B}$ the geodesic $\gamma_r(s)$ as defined in the proof of Theorem 13 satisfies this theorem. We can write $\mathcal{B}$ as:

$$\mathcal{B} = \{(p,r) \in E : v_s(p,r) \neq 0 \quad \forall s \in \mathbb{R}\}$$

Theorem 18 *Let $f\colon M^n \to \mathbb{R}^n$ be a continuous map. Then $\mathcal{B}$ has at least dimension $n-1$.*

Proof We use the same construction as in Theorem 13. Let f_n have its maximum in p. Take an arbitrary element $r \in S(p)$, then by the same argument as in Theorem 13 we must have a t between $s = 0$ and $s = -\frac{1}{2}$ such that $v_t(p,r) = 0$, so we have $f(\gamma_r(t)) = f(\gamma_r(t+a))$. Because this holds for each $r \in S(p)$ we have that $S(p) \subset \mathcal{B}$ and with dim $S(p) = n-1$ we come to the end of the proof. □

5 Real-valued mapping of the sphere

In this section we use the notation $\overset{\circ}{B}$ for the interior and $\overline{B}$ for the closure of the set B. For the theorem by Floyd we give the following definitions.

Definition 21 (separation of a set) For any set M, a representation $M = M_1 \cup M_2$ of M as the union of two nonempty disjoint open sets is called a **separation**. The sets M_1 and M_2 in a separation are called separated sets.

Definition 22 (connected) A set is said to be **connected** if and only if it admits no separation whatever.

Definition 23 (component) A **component** of a topological space X is the union of all connected subsets of X which contain a certain fixed point $x \in X$.

A component is not a real subset of a larger component.

Definition 24 Let X be a topological space. A subset A of X **separates** X if $X \backslash A$ has more than one component. Two points of X are said to be separated by a subset A of X if they lie in different components of $X \backslash A$.

Lemma 19 *If Y is a subspace of X, a separation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other, i.e. $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$. The space Y is connected if there exists no separation of Y.*

Definition 25 (unicoherent) A connected space is **unicoherent** provided that, no matter how it is represented as the union of two closed, connected sets, the intersection of these sets is connected.

Examples of unicoherent spaces are a arc and the 2-disk. Spaces which are not unicoherent are for example the circle and the torus.

Definition 26 (locally connected) A space X is said to be **locally connected** at a point $x \in X$ if for every neighborhood U of x, there is a connected neighborhood V of x contained in U. If X is locally connected at each of its points, it is simply said to be locally connected.

Lemma 20 *A space X is locally connected if and only if for every open set U of X, each component of U is open in X. Then for any subset A of X which separates X the space $X \backslash A$ has a separation.*

Definition 27 (continuum) A **continuum** is a compact connected set.

Definition 28 (map of period n) A **map of period** n is map of which the n^{th} composition of this map with itself is the identity map.

In order to prove Theorem 22 of Floyd we need the following lemma.

Lemma 21 (Floyd) *Suppose that X is a unicoherent locally connected continuum, and that T is a map of period 2 on X without fixed points. Suppose A is a subset of X which*
(i) is closed in X,
(ii) is invariant under T and
(iii) separates x from $T(x)$ for all $x \in X \backslash A$.
Then there exists a connected subset B of A having properties (i), (ii) and (iii).

PROOF There exists a set $B \subset A$ minimal with respect to possessing properties (i), (ii) and (iii). Then by (iii) we have that $X \backslash B$ is separable by T (, i.e. B separates X into the components T and $T(P)$), so $X \backslash B = P \cup T(P)$ with $P \cap T(P) = \emptyset$ and $P, T(P) \neq \emptyset$.
Note that because B is minimal we only have two components and because X is locally connected they are open by Lemma 20. We denote $Q = T(P)$ and because T has period 2 we have $P = T \circ T(P) = T(Q)$. Because B is minimal we have that P is maximal with respect to the class to separable sets, i.e. sets R with $\overline{R} \cap T(R) = R \cap \overline{T(R)} = \emptyset$. $X \backslash B$ is open because B is closed. Because P and $T(P)$ are open sets in $X \backslash B$ they are open sets in X.
We now want to show that $\mathring{B} = \emptyset$. Arguing by contradiction, suppose $\overline{B} \neq \emptyset$. Then for

$x \in \overset{\circ}{B}$ we have $x \notin P \cup Q$ and then $(\overset{\circ}{B} \cup P) \cap Q = \emptyset$. This is in contradiction with the maximality property of P. Then the closure of $P \cup Q$ is the whole space X and because P and Q are open sets in X we have that $(\partial P \cup \partial Q) \cap (P \cup Q) = \emptyset$ and then by the separation $X \backslash B = P \cup Q$ it follows that $(\partial P \cup \partial Q) = B$. Thus

$$\overline{P} \cap \overline{Q} = \partial P \cup \partial Q = B.$$

We claim that $\overline{P}$ is connected and by continuity if T it follows that $\overline{Q} = T(\overline{P})$ is connected, so that we have proved the conclusion of the lemma. We will prove the claim by contradiction, i.e. we assume that $\overline{P} = U \cup V$ is a separation. Then we have $\overline{Q} = T(U) \cup T(V)$. We get the following separation

$$\begin{aligned} B &= \partial P \cup \partial Q \\ &= (U \cup V) \cap (T(U) \cup T(V)) \\ &= (U \cap T(U)) \cup (V \cap T(V)). \end{aligned} \tag{10}$$

Since $B = (U \cap B) \cup (V \cap B)$ is also a separation we have

$$\begin{aligned} U \cap T(U) &= U \cap B, \\ V \cap T(V) &= V \cap B. \end{aligned}$$

Then from

$$(U \cap B) \cup (U \cap T(V)) = (U \cap T(U)) \cup (U \cap T(V)) = U \cap \overline{Q} = U \cap B,$$

it follows that $(U \cap T(V)) \subset (U \cap T(U))$. But because (10) is a separation we have that a nonempty subset of $T(V)$ is not contained in $(U \cap T(U))$, so $(U \cap T(V)) = \emptyset$. In a similar way $(U \cap T(V)) = \emptyset$ and we get the following separation of X,

$$X = \overline{P} \cup \overline{Q} = U \cup T(U)) \cup (V \cup T(V),$$

which is in contradiction with the connectivity of X. $\square$

Theorem 22 (Floyd) *Let $f\colon S^2 \to \mathbb{R}$ be a continuous map and let $x_0, x_1, x_2 \in S^2$. There exists a rotation ρ of S^2 with $f(\rho(x_0)) = f(\rho(x_1)) = f(\rho(x_2))$.*

Proof Let $SO(3)$ denote the group of rotations of S^2, which is homeomorphic to $\mathbb{R}P^3$ and a unicoherent locally connected continuum. Let s_i be a rotation of order 2 interchanging x_0 and x_i, $i = 1, 2$. Define $T_i\colon SO(3) \to SO(3)$ by $T_i(\rho) = \rho \circ s_i$. Define mappings $f_i\colon SO(3) \to \mathbb{R}$ by $f_i(\rho) = f(\rho(x_i)) - f(\rho(x_0))$, so that $f_i \circ T_i = -f_i$. Define the closed set $A_i = \{\rho \in SO(3) | f_i(\rho) = 0\}$, which is invariant under T_i. For $\rho \in X \backslash A_i$, the points $f_i(\rho)$ and $f_i(T(\rho))$ have opposite signs and therefore A_i separates ρ and $T(\rho)$. Then we can use Lemma 21 and we have connected subsets B_i of A_i. The transformations preserve Haar measure in $SO(3)$. So if U is a nonempty open set in $SO(3)$ with $U \neq SO(3)$ then $T_1(T_2(\overline{U})) \not\subseteq U$, otherwise it would contradict the measure preserving property. Suppose $B_1 \cap B_2 = \emptyset$. We have the separations $X/B_1 = P_1 \cup P_1$ and $X/B_2 = P_2 \cup P_2$. With B_i

connected, we can say that $B_1 \subset P_2$ and $B_2 \subset Q_1$, so $B_2 \cap P_1 = \emptyset$. From this and $B_1 \cap B_2 = \emptyset$ we get $(P_1 \cup B_1) \cap B_2 = \emptyset$ and therefore

$$(P_1 \cup B_1) \subset SO(3) \backslash B_2 = P_2 \cup Q_2. \tag{11}$$

Note that $P_1 \cup B_1$ is connected, because B_1 is a continuum and $\partial P_1 \subset B_1$. Together with $B_1 \subset P_2$ and $P_2 \cap Q_2 = \emptyset$ it follows that $P_1 \cup B_1 \subset P_2$. Similarly $Q_2 \cup B_2 \subset Q_1$. We also have seen in the proof of the lemma that $\overline{P}_1 \subset P_1 \cup B_1$. Then we get:

$$T_1(T_2(\overline{P}_1)) \subset T_1(T_2(P_1 \cup B_1)) \subset T_1(T_2(P_2)) \subset T_1(Q_2) \subset T_1(Q_1) = P_1, \tag{12}$$

which contradicts the measure preserving property. Hence $B_1 \cap B_2 \neq \emptyset$. For $r \in B_1 \cap B_2$ we have:

$$f_1(\rho) = f_2(\rho) = 0 \text{ then } f(\rho(x_0)) = f(\rho(x_1)) = f(\rho(x_2)). \tag{13}$$

□

6 Counterexamples to Knaster's conjecture

We will denote $\{x_i\}_{i=1}^{n+1}$ for the coordinate functions of the unit sphere S^n in R^{n+1}. For $j < n$ we define the subsphere of S^n as $S^j = \{s \in S^n : x_i(s) = 0 \text{ for } i > j+1\}$. For example for $j = 1$ we have $S^j = S^1$ a great circle in S^n.
We define:

$$f_s = \frac{1}{2}(f(s) + f(-s))$$
$$f_a = \frac{1}{2}(f(s) - f(-s))$$

We can write a continuous map $f\colon S^n \to \mathbb{R}^m$ as

$$f(s) = \frac{1}{2}(f(s) + f(-s)) + \frac{1}{2}(f(s) - f(-s)) = f_s + f_a.$$

We decompose the map f into a symmetric part f_s and anti-symmetric part f_a by setting $f = f_s + f_a$. A map f_s is symmetric if $f_s(x) = f_s(-x)$. A map f_a is anti-symmetric if $f_a(x) = -f_a(-x)$.
In the next counterexample Chen uses this decompostion when dealing with antipodal points in the configuration. When we have both s_1 and $-s_1$ in our configuration, so by Knaster Conjecture and the fact that rotation preserves sign, we have $f(\rho(s_1)) = f(\rho(-s_1)) = f(-\rho(s_1))$. By the property of anti-symmetry we then have $f_a(\rho s_1) = f_a(-\rho s_1) = 0$.
To give an indication from where the bound on k in Knaster's conjecture come form, we will prove that for $k > n - m + 2$ a counterexample is easy to find.

Lemma 23 *For $k = n - m + 3, k > 1$ points there exists a continuous map $f\colon S^n \mapsto \mathbb{R}^m$ and a special configuration of k points $s_1, \ldots, s_k \in S^n$ for which the conclusion of Knaster's conjecture is false.*

PROOF We will proof this lemma by contradiction in two cases: $m \geq n+1$ and $0 < m < n+1$. For both cases we let f be the first m coordinate functions. We write e_i, with $i \in \{1, \ldots, n+1\}$, for the standard basis vectors in $\mathbb{R}^{n+1}$.
Case 1. We have $f = (x_1, \ldots, x_m)$ and $k = 2$. For $s_1 \neq s_2$ it follows that $\rho(s_1) \neq \rho(s_2)$, so $x_i(\rho(s_1)) \neq x_i(\rho(s_2))$ for some coordinate function x_i.
Case 2. We have $f = (x_1, \ldots, x_m)$. Let the k point configuration be:

$$\begin{aligned} s_1 &= e_1, \\ &\vdots \\ s_{k-1} &= e_{k-1}, \\ s_k &= -e_{k-1}. \end{aligned}$$

Then the first k points on S^n in $\mathbb{R}^{n+1}$ span a $(k-1)$-dimensional subspace of $\mathbb{R}^{n+1}$ and therefore under a rotation, ρ, the vectors $\{\rho(s_i)\}_{i=1}^k$ also span a $(k-1)$-dimensional subspace. The map f is anti-symmetric, so $f(\rho(s_{k-1})) = -f_\rho(s_{k-1}) = 0$. Then in order for Knaster's conjecture to be true we need $f(\rho(s_i)) = 0$ for $i \in \{1, \ldots, k\}$. This means that the vectors $\{\rho(s_i)\}_{i=1}^k$ are in the space spanned by the coordinate functions $x_{m+1}, \ldots, x_{n+1}$, which has dimension $n - m + 1 = k - 2$. This is in contradiction to the fact that the vectors $\{\rho(s_i)\}_{i=1}^k$ must span a $(k-1)$-dimensional subspace. □

Counterexample 1 Let $n = 6$, $m = 3$, $k = 5$. Let $\{s_i\}_{i=1}^4$ be points chosen on S^1 such that no two points are antipodal. Let

$$f = (max\{|x_1|, |x_2|\}, max\{|x_3|, |x_4|\}, max\{|x_5|, |x_6|\})$$

This gives a counterexample to Knaster's Conjecture

PROOF Proof by contradiction, so suppose there exists a rotation $\rho \in SO(7)$ satisfying Knaster's Conjecture. The great circle $C = S^1$ cannot be rotated by ρ such that it lies entirely on the x_7-axis, so it must have a nontrivial projection onto one of the other axes, say x_1. Then we have

$$f(\rho(s_1)) = \cdots = f(\rho(s_5)),$$

so that for the first coordinate we have

$$f_1(\rho(s_1)) = \cdots = f_1(\rho(s_5)),$$

and thus

$$max\{|x_1(\rho(s_1))|, |x_1(\rho(s_1))|\} = \cdots = max\{|x_1(\rho(s_5))|, |x_1(\rho(s_5))|\}. \tag{14}$$

We can write $x_1(\rho(s_i)) = cos\theta_i$ for four distinct angles θ_i and where r is the maximum of the projection of C on the x_1-axis. Since no two points of $s_1, \ldots, s_5$ are antipodal, we have $\theta_i \neq \theta_j + \pi$, hence we have at most two angles θ_i, such that $x_1(\rho(s_i)) = rcos\theta_i$. The same holds for x_2. That means that there are at most two points, say s_1, s_2 on S^1, such that $x_1(\rho(s_1)) = x_1(\rho(s_2)) = c$ and also at most two points, say s_3, s_4 on S^1, such that $x_2(\rho(s_3)) = x_2(\rho(s_3)) = \pm c$. Then there are at most four points s_i such that $max\{|x_1(\rho(s_i))| = c$, contradicting condition (14) □

Counterexample 2 Suppose $n \leqq mk'$, where $k' = \frac{k-1}{2}$. Then we can construct a counterexample by $\{x_i\}_{i=1}^k$ on S^1 such that no two point are antipodal. Let f be such that

$$f(x_1, \dots, x_n) = max\{|x_{(j-1)k'+1}|, \dots, |x_{jk'}|\}$$

where x_i is understood to be 0 for $i > n + 1$.

PROOF This conjecture is the general case of the previous conjecture and thus we use the same arguments. There must be some x_i for $i \leq n$, such that $C = S^1$ has non-trivial projection. The function is the maximum of k' coordinate functions, so this maximum, $f_j = c > 0$, can only be achieved at $2k'$ points of the k points $s_1, \dots s_k$. Since $k = 2k' + 1$, we reach a contradiction. $\square$

Counterexample 3 Let $n > 2$, $m = n - 1$, $k = 3$. Let $s_1 = (1, 0, 0, \dots, 0)$, $s_2 = -s_1$ and $s_3 = (0, 1, 0, \dots, 0)$. Let $f_a = \epsilon(x_3, \dots, x_{n+1})$, where $\epsilon = 0.1$.
Let $f_s = (x_1, x_2, (x_1^2 - x_2^2)/2, 0, 0, \dots)$. If we set $f = f_s + f_a$, then this produces a counterexample of Knaster's Conjecture.

PROOF Proof by contradiction, so suppose there exists a rotation $\rho \in SO(6)$ satisfying Knaster's Conjecture and thus $f_1(t_1) = f_1(t_3)$ holds, with $t_i = \rho s_i$. Then by anti-symmetry of f_a, we have $f_a(t_1) = f_a(t_2) = 0$ and thus $t_1 \in S^1$. We note that rotations preserve angles so $t_1 \cdot t_3 = s_1 \cdot s_3 = 0$. We will construct this counterexample by proving three claims.
We will first begin with the proof of claim 1: $f_1(t_1) = |x_1(t_1)x_2(t_1)| < \epsilon$.
Then we will proof claim 2: $|f_2(t_1)| < \epsilon$.
At last we will proof claim 3: we obtain a contradiction between claim 1 and claim 2.
We will proof claim 1 and claim 2 by contradiction.

The proof of claim 1.

Remark 4 If $x_1(t_1)x_2(t_1) > 0$ then by

$$t_1 \cdot t_3 = x_1(t_1)x_1(t_3) + x_2(t_1)x_2(t_3) = 0$$

we get $x_1(t_3)x_2(t_3) < 0$ and for $x_1(t_1)x_2(t_1) < 0$ we have, by the same equation, $x_1(t_3)x_2(t_3) > 0$.

Let us take the case where $x_1(t_1)x_2(t_1) > \epsilon$, then we have by Remark 4 that $x_1(t_3)x_2(t_3) < 0$. So we must have

$$f_1(t_3) = x_1(t_3)x_2(t_3) + \epsilon x_3(t_3) \leq \epsilon, \tag{15}$$

since $x_3(t_3) \leq 1$. We already have $f_1(t_1) = x_1x_2 > \epsilon$, so we obtain a contradiction to $f_1(t_1) = f_1(t_3)$. For $x_1x_2 < -\epsilon$ we have by Remark 4 that $x_1(t_3)x_2(t_3) > 0$ and so by equation (15) we have $f_1(t_3) \geq -\epsilon$. Then again we obtain a contradiction. With both contradictions we prove claim 1.

The proof of claim 2.
We will proof the claim $|f_2(t_1)| < \epsilon$ by contradiction, i.e. we assume that $|f_2(t_1)| \geq \epsilon$.

We will begin with the positive case: $f_2(t_1) \geq \epsilon$.
Because $t_1 \in S^1$ we have $|x_1(t_1)| > |x_2(t_1)|$ and from

$$t_1 \cdot t_3 = x_1(t_1)x_1(t_3) + x_2(t_1)x_2(t_3) = 0, \tag{16}$$

it follows that $|x_1(t_1)||x_1(t_3)| = |x_2(t_1)||x_2(t_3)|$, so that $|x_2(t_3)| \geq |x_1(t_3)|$ and thus $x_1(t_3)^2 + x_2(t_3)^2 \leq 0$. We then have

$$f_2(t_3) = \frac{x_1(t_3)^2 + x_2(t_3)^2}{2} + \epsilon x_4(t_3) \leq \epsilon$$

and obtain a contradiction to the assumption that $f_2(t_1) = f_2(t_3)$.
For the negative case we assume $f_2(t_1) < -\epsilon$, so that we have $|x_2(t_1)| > |x_1(t_1)|$ and thus again by (16) we have $|x_1(t_3)| \geq |x_2(t_3)|$, so $f_2(t_3) \geq -\epsilon$ and we obtain a contradiction.

The proof of claim 3.
Note that:

$$(x_1 + x_2)(x_1 - x_2) = x_1^2 - x_2^2,$$

so that from claim 2 it follows

$$|f_2(t_1)| = |\frac{(x_1(t_1) + x_2(t_1))(x_1(t_1) - x_2(t_1))}{2}| \leq \epsilon.$$

Then we have one of the following cases,
case 1,

$$0 \leq \frac{(x_1(t_1) + x_2(t_1))(x_1(t_1) - x_2(t_1))}{2} \leq \epsilon$$

or case 2,

$$-\epsilon \leq \frac{(x_1(t_1) + x_2(t_1))(x_1(t_1) - x_2(t_1))}{2} \leq 0.$$

Note also that:

$$\frac{x_1^2(t_1) + x_2^2(t_1)}{2} = \frac{(x_1(t_1) + x_2(t_1))(x_1(t_1) - x_2(t_1))}{2} + x_2^2(t_1) = \frac{1}{2},$$

so that we have

$$x_2^2(t_1) - \frac{1}{2} = \frac{(x_1(t_1) + x_2(t_1))(x_1(t_1) - x_2(t_1))}{2}.$$

Then from case 1 and case 2 we get one of the following two cases,
case 1

$$\frac{1}{2} \leq x_2^2(t_1) \leq \frac{1}{2} + \epsilon$$

or case 2

$$\frac{1}{2} - \epsilon \leq x_2^2(t_1) \leq \frac{1}{2}.$$

With $x_1^2(t_1) + x_2^2(t_1) = 1$ we then get from case 1,

$$\frac{1}{2} - \epsilon \leq x_1^2(t_1) \leq \frac{1}{2}$$

or from case 2,

$$\frac{1}{2} \leq x_1^2(t_1) \leq \frac{1}{2} + \epsilon.$$

Finally we get from both cases the following inequality:

$$\frac{1}{4} - \frac{1}{2}\epsilon \leq x_1^2(t_1)x_2^2(t_1) \leq \frac{1}{4} + \frac{1}{2}\epsilon.$$

From claim 1, we have $x_1^2(t_1)x_2^2(t_1) \leq \epsilon^2$. Then for $\epsilon = 0.1$, we obtain a contradiction. □

Remark 5 The arguments used in the proof of this counterexample can't be extended to the unknown cases, because we need $t_1 \in S^1$, thus $n - m = 1$.

Counterexample 4 Let $n = 5$, $m = 3$, $k = 4$. Let the configuration of points $\{s_i\}_{i=1}^4 \in S^1$ be

$$s_1 = (1, 0),\ s_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2}),\ s_3 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}),\ s_4 = -s_1.$$

Let $f_a = \epsilon(x_4, x_5, x_6)$, where $\epsilon = 0.1$.
Let $f_s = (|x_1|, |x_2|, |x_3|)$ and then for $f = f_s + f_a$ we have a counterexample of Knaster's Conjecture.

Proof Proof by contradiction, so suppose there exists a rotation $\rho \in SO(6)$ satisfying Knaster's Conjecture. Let $t_i = \rho s_i$, then, by the property of anti-symmetry of f_a, $f_a(t_1) = 0$ and thus $t_1 \in S^2$. We will show that $f_1(t_1) = |x_1(t_1)| < 2\epsilon$ by contradiction, i.e. suppose $c = |x_1(t_1)| \geq 2\epsilon$. By the assumption of Knaster's conjecture we have $f_1(t_1) = f_1(t_i)$, from which it follows:

$$c = |x_1(t_1)| = f_1(t_1) = f_1(t_i) = |x_1(t_i)| + \epsilon x_4(t_i),$$

for $i = 2, 3$. Since $|x_4(t_i)| \leq 1$, we get $c - \epsilon \leq |x_1(t_i)| \leq c + \epsilon$. Note that $s_1 + s_3 = s_2$, then we have

$$x_1(t_1) + x_1(t_3) = x_1(\rho(s_1 + s_3)) = x_1(\rho(s_2)) = x_1(t_2).$$

Hence $|x_1(t_2)| = |(|x_1(t_1)| \pm |x_1(t_3)|)|$. So we have either $|x_1(t_2)| \geq 2c - \epsilon \geq c + \epsilon$ or $|x_1(t_2)| \leq \epsilon \leq c - \epsilon$ and this is in contradiction to $c = |x_1(t_1)|$. In the same way we must have $c = |x_2(t_1)| \geq 2\epsilon$ and $c = |x_3(t_1)| \geq 2\epsilon$, so

$$x_1^2(t_1) + x_2^2(t_1)x_3^2(t_1) < 12\epsilon^2 < 1.$$

Then $t_1 \notin S^2$, which gives a contradiction. □

Remark 6 The arguments used in the proof of this counterexample can't be extended to the unknown cases. To let a similar argument work we need at least $m = 3$ for having

$$x_1^2(t_1) + x_2^2(t_1) + x_3^2(t_1) < 1,$$

so we need $t_1 \in S^2$ and $m > 2$ in order to get a contradiction. But in the unknown cases of Knaster's conjecture we have $m \leq 2$.

Chen gives in [3] his own conjecture:

Conjecture 2 (Chen) Knaster's conjecture is false for $n > m > 1$.

We will proof in the next lemma that in a special case, Knaster's conjecture is true.

Lemma 24 *Let $k = 4$, then $m = n - 2$. Let the configuration of points, $\{s_i\}_{i=1}^4 \in S^1$, contain two pairs of antipodal points, i.e. $s_1 = -s_3$, $s_2 = -s_4$. For any symmetric continuous map, $f = f_s$, there exist a rotation, $\rho \in SO(n+1)$, such that*

$$f\rho(s_1) = \cdots = f\rho(s_4).$$

Proof We define $\mathbb{R}P^n = S^n/\sim$, with $x \sim y$ if and only if $y = \pm x$. Let $p \colon S^n \mapsto \mathbb{R}P^n$ be the projection map by sending $x, -x \in S^n$ to the corresponding equivalence class $[x] \in \mathbb{R}P^n$. Restrict the map f to a subsphere S^{n-2} of S^n,

$$f|_{S^{n-2}} \colon S^{n-2} \to \mathbb{R}^{n-2}.$$

We can define a continuous map $g \colon \mathbb{R}P^n \mapsto \mathbb{R}^{n-2}$; by $g([x]) = f(|_{S^{n-2}}(x) = f|_{S^{n-2}}(-x)$, so that $f|_{S^{n-2}} = g \circ p$. By Hopf's Theorem 13, we can find two distinct points, $[x_1]$, $[x_2] \in RP^{n-2}$, such that $g([x_1]) = g([x_2])$ and $d_{S^{n-2}}(x_1, x_2) = d_{S^{n-2}}(s_1, s_2)$, with $d_{S^{n-2}}$ the spherical distance. From $g([x_1]) = g \circ p(x_1)) = f|_{S^{n-2}}(x_1) = f(x_1)$ we get $f(x_1) = f(-x_1) = f(x_2) = f(-x_2)$. When we take the spherical distance between x_1 and x_2 to be the same spherical distance between s_1 and s_2 we can find a rotation $\rho \in SO(n)$ such that $\rho(s_i) = x_i$ for $i = 1, 2$. By the fact that $\rho(-s_i) = -\rho(s_i)$ we prove the lemma. □

This lemma can be applied to some of the unknown cases of $k = 4$, namely $n = 3$, $m = 1$ and $n = 4$, $m = 2$.

7 Discussions and Conclusions

We proved for the special case of symmetric functions and two pairs of antipodal point that for $k = 4$, $n = 3$, $m = 1$ and $n = 4$, $m = 2$ Knaster's conjecture holds. An approach to the problem is to combine several results, in the way we did in Lemma 24, for example the Theorems of Floyd 22 and Hopf 13 together. If we look at the case $n = 3$, $m = 1$ and $k = 4$ then by Hopf 13 we can get for every circle or sphere in S^3 a pair of points, such that they are mapped to the same value. We need to get a grip on the value of that point obtained by f, in order to construct a counterexample or proof of the affirmation of this case.

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