The Reconstruction Conjecture

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In the early 1900's, two great mathematicians made a conjecture about the reconstructibility of a graph. Specifically they said this:

The Kelly-Ulam Conjecture

Let G and H be graphs with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and $V(H) = \{u_1, u_2, \cdots, u_n\}$, for $n \geq 3$. If $G - v_i = H - u_i, \forall i = 1, \cdots, n$, then $G = H$.

We have not yet been able to prove the conjecture true or false. The work that has been done with the conjecture has helped us gain a better understanding about what kinds of graphs are always reconstructable and also what kinds of graphs we do not yet know about.

We say that a graph G is *reconstructable* if we can recover the unique graph, up to isomorphism, from the vertex deleted subgraphs. This says that given $G - v_i, \forall i = 1, \cdots, n$, we can recover G or a graph that is isomorphic to G.

Frank Harary has restated the Kelly-Ulam Conjecture using this definition. Sometimes it is more helpful to look at his restatement rather than the original reconstruction conjecture. It does no matter which version of the conjecture that we are considering because the two are equivalent.

Harary's Restatement of the Reconstruction Conjecture

All graphs of order three or more are reconstructable.

Now, let us look at some of the basic information about graphs that we can use in reconstruction.

Basic Information

The reconstruction conjecture is only stated for graphs of order 3 or more. We already know that if G and H have order 2, then the reconstruction conjecture is false.

A graph G is referred to as *labeled* if its vertices are associated with distinct labels in a one to one manner. We refer to the labels as the vertex set of G. However, in consideration of the reconstruction conjecture, we do not consider that the vertex deleted subgraphs are labeled when we are looking at them. If the vertex deleted subgraphs were labeled, then the reconstruction conjecture would be trivially true.

Example

Let H be the graph consisting of just two vertices and G be a path of order 2. Then it is obvious that $G - v_i$ is isomorphic with $H - u_i$ for $i = 1, 2$. However, with the H and G that we started with, the two graphs are not isomorphic to each other.

$G =$ 1 ———— 2 $\quad H =$ 1 $\qquad$ 2

We refer to the graphs $G - v_i, \forall i = 1, \cdots, n$ as the vertex deleted subgraphs. Denote the graph $G - v_i = G_i$. If we have the collection of vertex deleted subgraphs there are several facts that we know that we recover about the original graph G. We refer to these items as *recoverable*. We will also refer to the set of all the vertices of G as $V(G)$.

One of the most obvious recoverable facts about G is its order. We know that the order of G is going to be at least three. If we have all of the G_i then we will have exactly one G_i for each of the vertices $v_i \in G$. From this we can see that the number of G_i that we have is going to be the same as the order of G. Since we know that the order of a graph G is recoverable, let n represent the order of a graph.

Another fact about G that is recoverable is the total number of edges, q. When looking at the graphs G_i we notice that each of the edges in G appears in $n-2$ of the subgraphs. Specifically the two G_i that any edge would not be in are the two G_i where the vertices associated with that specific edge are the deleted vertices. Let q_i be the number of edges in each of the $G_i, i = 1, \cdots, n$, then the total number of edges is

$$q = \sum \frac{\mathrm{q_i}}{\mathrm{n-2}}.$$

We are also able to tell the degree of each $v_i \in G$. Now that we know that there are p edges in the whole graph G and from the way that we have constructed the subgraphs G_i, we can see that in each of these G_i the only edges that are missing are going to be the ones that have v_i as an endpoint. Therefore,

$$degv_i = q - q_i, \forall i = 1, \cdots, n.$$

In a connected graph G, we say that v is a *cut vertex* if the graph $G - v$ is disconnected. In a disconnected graph G, v is a *cut vertex* if it is a cut vertex in any of the components of G.

Theorem 1

Let G be a non-trivial connected graph and let $u \in V(G)$. If v is a vertex of G such that the length of a $u - v$ path in G is maximal, then v is not a cut vertex.

Proof:

Assume that v is a cut vertex. Then, there exists a vertex w of $G - v$ such that w is in a different component of $G - v$ than u. Then because v is a cut vertex in G, every $u - w$ path contains v. Therefore there is a $u - w$ path which has length longer than our maximal $u - v$ path. This is a contradiction, so v must not be a cut vertex.$_{//}$

Corollary 1

If G is a non-trivial connected graph, then G contains at least two vertices that are not cut vertices.

Proof:

Let $u, v \in V(G)$ such that the length of a $u - v$ path in G is maximal. Then by Theorem 1, u and v are not cut vertices.$_{//}$

Corollary 2

If G is a graph of order $n \geq 3$ with q edges and G has no isolated vertices, then

a) if n is odd, $q \geq ((n+1)/2)$

b) if n is even, $q \geq (n/2)$

Proof by induction:

Start with $n = 3$

Since there are no isolated vertices, then each vertex must have an edge from one of the other vertices to it. By inspection, the least number of edges that you could have then is 2.

$2 \geq (3+1)/2 = 2$

So the inequalities hold for $n = 3$

Next show that if the inequalities hold for n and then they hold for $n+1$.

Case 1, n is even

Then let q_i be the number of edges in a graph of order i. Assume that $q_n \geq (n/2)$. We need to show that $q_{n+1} \geq (((n+1)+1)/2)$. To get a graph of order $n+1$ from any graph of order n, we have to add a vertex. Since we have assumed that there are no isolated vertices, we also have to add at least one edge to connect that vertex to another vertex already in the graph. Then it follows that

$$q_{n+1} \geq q_n + 1 \geq (n/2) + 1 = (n+2)/2 = ((n+1)+1)/2.$$

So from this we get that $q_{n+1} \geq (((n+1)+1)/2)$.

Case 2, n is odd

Assume that $q_n \geq ((n+1)/2)$. We need to show that $q_{n+1} \geq ((n+1)/2)$. Since there are no isolated vertices in G, then by the same argument as in case 1,

$$q_{n+1} \geq q_n \geq ((n+1)/2).$$

So from this we get that $q_{n+1} \geq ((n+1)/2)$.

Then by induction the corollary is true for all $n \in \mathbf{N}$.//

Theorem 2

If G is a graph with $V(G) = \{v_1, \cdots, v_n\}$, for $n \geq 3$, and G_i is the subgraph with the vertex v_i and the edges incident to it deleted $\forall i = 1, \cdots, n$. Then

G is connected iff at least two of the G_i are connected.

Proof:

Let G be connected. Then by Corollary 1, we know that G contains two vertices which are not cut vertices.

Assume that there exist two vertices of G, u, v, such that both $G - u$ and $G - v$ are connected. This means that in $G - u$, v is connected to each $v_i, i \geq 3$ and in $G - v$, u is connected to each $v_i, i \geq 3$. Then there exists a vertex $w \in V(G)$ such that there is a $u - w$ path in $G - v$ and there is a $v - w$ path in $G - u$. Then we know that there exists a $u - v$ walk in G. Therefore, by previous work we know that there exists a $u - v$ path in G. So, G is connected.

From Theorem 2, it is obvious that if we have all of the vertex deleted subgraphs, G_i, then we are able to tell if the graph is connected or disconnected. The connectivity of a graph is another fact about G that is recoverable from the vertex deleted subgraphs.

If G is connected then by looking at the G_i we can tell also how many cut vertices the original graph G has. We already know that we can tell if G is connected or not. If G is connected then we look at the number of the G_i which are disconnected. Each vertex v_i that created a disconnected G_i is a cut vertex.

Another fact about G that is recoverable is whether or not G is unicyclic. A connected graph G is *unicyclic* if it contains exactly one cycle.

Theorem 3

A (n, q) graph G is unicyclic if and only if G is connected an $n = q$.

Proof:

Let G be a (n, q) unicyclic graph and let e be an edge of the cycle of G. The $(n, q-1)$ graph $G - e$ is a tree and is connected. Then because the number of edges in a connected tree is equal to the number of vertices minus one. Therefore, $n - 1 = q - 1$. Obviously if G is unicyclic, it is connected based off of the definition of unicyclic. Also if $n - 1 = q - 1$ then it follows that $n = q$.

Let G be a connected (n, q) graph such that $n = q$. Since $n = q$, G is not a tree. Then because G is not a tree, not every edge of G is a bridge. Since there is some edge of G that is not a bridge, call it e, then e must be an edge on a cycle. Then since e is on a cycle, $G - e$ is connected and has $n - 1$ edges. Therefore, $G - e$ is a tree. So, e was an edge on the only cycle in G. Therefore, G is unicyclic.$_{//}$

Since we know that we can determine whether a graph G is disconnected or not and because we can recover the n and q. Then we can also determine from Theorem 3 whether a graph G is unicyclic or not.

A nontrivial connected graph G is *Eulerian* if it contains a circuit which touches every edge of G exactly one time.

Theorem 4

Let G be a graph in which every vertex has degree at least 2. Then G contains a cycle.

Proof:

Let v_0 be a vertex of G. Since it has degree at least 2, then we can pick on edge of G which has v_0 as one end and a vertex, v_1, as the other. Since the degree of v_1 is at least 2, we can pick an edge which has v_1 as one end and a vertex, v_2, as the other. We can continue this process until eventually we pick a vertex that has been chosen before. Thus, the choice of vertices that we have done thus far is a cycle of G.$_{//}$

Theorem 5

A connected graph G is Eulerian if and only if every vertex has even degree.

Proof:

Let G be a connected graph. Assume that G is Eulerian. Let C be an Eulerian circuit in G. Put u as the start and end vertex of C. Let v be a vertex of G such that $v \neq u$, then because G is connected, v is a vertex on C. Then, every time v is on the circuit, it must be we must use a different edge to come and leave v since C uses each edge of G exactly one time. Then, every time that v appears in C, we add two to its degree. Thus, v has even degree. Since

C began and ended at u, the beginning and ending edges of C add two to the degree of u. Any other time that u appears in C the same idea as v applies. Thus, the degree of u is even.

Let G be a connected graph such that every vertex has even degree.

Proof by induction on the number of edges in C.

First since G is connected if C has zero edges, then G must contain just the vertex u. Therefore, trivially, G is Eulerian.

Next assume that if C has $1, \cdots, n$ edges then G is Eulerian. We need to show that if C has $n+1$ edges, then G is Eulerian. Since G is connected, then no vertex of G has degree zero. Then by theorem 4, because every vertex has even degree, G contains a cycle. Call the cycle C_1. If this cycle contains all the edges of G then we are done. It C_1 does not contain every edge of G, then we delete every edge of G that we have already used. Thus creating a subgraph of G, call it G_1. We should note that G_1 can be disconnected and that G_1 has the same vertex set as G. Also, all the vertices of G_1 have even degree still since the edges we removed took away two edges from every vertex that had any edges removed from it. Also, obviously, G_1 has less than $n+1$ edges, so by the induction hypothesis, each of the components of G_1 must be Eulerian. Then because we obtained G_1 by deleting edges of G, each component of G_1 must have at least one vertex in common with C. Now, to obtain an Eulerian circuit, we start at any vertex of C and travel around C until we reach a vertex that is part of a non-empty component of G_1. When we reach such a vertex, we follow the Eulerian circuit around that component. Then we resume traveling around C until we reach the next such vertex. We continue this process until we arrive back at the vertex of C that we chose to start with. Thus we have constructed a Eulerian circuit in G.

Then by induction, for any connected graph G, if the degree of every vertex of G is even, then G is Eulerian.$_{//}$

We already know that we can recover the degree sequence of all the vertices of G. Therefore, it then follows from Theorem 4 that whether a graph is Eulerian or not is a recoverable fact about G.

Before we look at graphs that we know are reconstructable, it is important to point out that just because we know whether a graph G is reconstructable or not does not mean that we have determined a special technique to reconstruct that graph. The discovery of such a technique is another problem in graph theory that if we were to solve it, then the Reconstruction Conjecture would be proven.

Some Reconstructable Graphs

There are many graphs that we know are reconstructable. We are going to look at a few of these graphs and the proofs that they are reconstructable. We will assume that all of our graphs are not multigraphs or digraphs, since there is a proof that the reconstruction conjecture is false for both of these types of graphs.

Regular

First, we know that regular graphs are reconstructable.

Proof:

Let G be a regular graph of order $n \geq 3$. And let G_i be the vertex deleted subgraphs. From the definition of these subgraphs, we know that each of them is going to be missing one of the vertices of G. Since G is regular then we also know what the degree of each of the vertices in G is going to be, call it r. Looking at any one of the G_i we can insert one more vertex, replacing the one that has been deleted. Now we replace edges from our inserted vertex to any of the $v - i$ who have degree $r - 1$ until all the vertices have degree r. Then, G is reconstructed.$_{//}$

Complete

From this it follows that complete graphs are reconstructable.

Proof:

Let G be a complete graph of order $n \geq 3$. Then by the definition of a complete graph, G is $n-1$ regular. Then since regular graphs as reconstructable, G is also.$_{//}$

Disconnected

We also know that disconnected graphs are reconstructable.

Proof:

We know from Theorem 2 that we can determine from the vertex deleted subgraphs whether the graph G is disconnected or not. Assume that G is disconnected. Then, at most 1 of its vertex deleted subgraphs are connected.

Let the order of G be $n \geq 3$. Put $V(G) = \{v_1, \cdots, v_n\}$. Also let q_i be the number of edges in each of the $G - v_i$. Also let q be the number of edges in G. We know from above that we can compute the values of the numbers q and p. We also know that we can compute the degree for each $v_i \in G$ and more specifically that we can associate that degree sequence with each of the v_i.

If for some j, $q_j = q$, then it is obvious that the vertex v_j is an isolated vertex in the graph G. So then the graph G consists of the vertex deleted subgraph $G - v_j$ with one added vertex, namely the vertex v_j.

Now, assume that G has no isolated vertices. If there are no isolated vertices, then it follows from Corollary 2, and the fact that for any $n, (n+1)/2 \geq n/2$, that $q \geq (n+1)/2$. Therefore, G cannot be trivial if the order of G is at least 3 and there are no isolated vertices. Then by Corollary 1, G contains at least two vertices which are not cut vertices.

Then for some G_i there will be a component F which contains the smallest number vertices. Let m be the number of vertices in this component. Then it is obvious that the deleted vertex v_i must be in this component F in the original graph G. If it was not in F then it would have to be in some other component F'. Then there would be a vertex deleted subgraph G_j with a component that would have $m-1$ vertices in it. This contradicts the assumption that m is the least number of vertices in any of the components of the subgraphs.

Now, because we are able to isolate the components that contain the deleted vertex in the way described above, then we will only consider subgraphs which have a component which is isomorphic to the original component F. In any of these graphs, label the components which have more than m vertices as $F_2, \cdots, F_k$. It is clear that these components are components contained in the the graph G. So we are only interested in the components in the subgraphs which are isomorphic to F since this is the only component of G that we have left to reconstruct.

We must consider 3 cases:

1. Some component $F_i, i = 2, \cdots, k$ has order at least $m + 3$.

   Let b denote the number of components which have order $m+1$. However, please note that b can be 0. Then if we select one subgraph G_j with k components such that $b + 1$ of these components have order $m + 1$. This means that v_j belongs in a component that has order greater than $m+2$. If v_j was not in a component of larger order, then it would be in a component with order $m + 1$ then there would be $b + 1$ components of order $m + 1$ which contradicts the definition of b. So then the $r + 1$ components that have order $m + 1$ are all components of G, one of them is F. Then the original G consists of these components which have order $m + 1$ in G_j together with any of the previously assigned $F_i, i = 2, \cdots, k$ which have order greater than $m + 1$. So G is reconstructed.

2. All components of $F_i, i = 2, \cdots, k$ have order $m + 2$.

   Look at all all the remaining G_i which have k components, two of which have order $m + 1$. Each time, one of those two components is F. If there is only one graph which appears in each pair, then it is F. If this is not the case then every pair of components will be two non-isomorphic components. Call these two F' and F''. One of either F' or F'' is F, the other was obtained by deleting a non-cut vertex from some component of G, specifically $F_i, i = 2, \cdots, k$. So look at the F_i. Pick one and remove a non-cut vertex from this component. You will then obtain a graph that is either F' or F''. Which ever one you do not get by removing the vertex is F.

3. At least one component among the $F_i, i = 2, \cdots, k$ has order $m + 1$, all others have order $m + 2$.

   First, some basic things to remember. If we are considering all components of G_i which have k components of order m, then all components that have order greater than m will be components of G. Also, a graph G' is a component of G iff it has order greater than m and it is a component of G_j for some j where G_j has at least one component of order m.

   Now, if each G_i has all but one of its components isomorphic to G' then every component of G is isomorphic to G'. If there is some components of the $G_i s$ which is not isomorphic to G', then notice that the number of components in G which are isomorphic to G' is going to be the same as the maximum number of components that are isomorphic to G', denote this maximum as c' in one of the G_i where one of the components has order m. If this is the case, then looking at the G_i which gave us this maximum we get that all the components of this G_i which are isomorphic to G' are components in G. Then we look at a G_j which has less than c' components isomorphic to G', specifically it will have $c' - 1$ components isomorphic to G'. Then v_j must have been deleted from one of the components that is isomorphic to G'. Then the other $k - c'$ components are also components of G. Then G has been reconstructed.

In this case there is one exception. If only one component has order $m+2$ and G' is a component of order $m+1$ and every component of order $m+1$ in each of the G_i is isomorphic to G' then the number of components of G which are isomorphic to G' is one more than the maximum that was previously described. However, this will not affect the reconstruction of G.

Therefore, disconnected graphs are reconstructable.$_{//}$

Compliment is Disconnected

From the above it follows that if the complement of G is disconnected then we can reconstruct G.

Assume that G is a connected graph such that the compliment of G, $\overline{G}$, is disconnected. Then from the G_i it is obvious that $\overline{G_i}$ is the same as taking the compliment of each of the G_i. Then when we have the compliment of each of the G_i we know that since $\overline{G}$ is disconnected that we can reconstruct it. From here, the definition of a compliment allows us to reconstruct G.$_{//}$

Trees

Another kind of graph that we know is reconstructable are trees. Before looking at that proof, let us introduce some terminology that will make the discussion easier.

A*tree* is a connected graph in which every edge is a bridge. An edge e in a connected graph G is called a *bridge* if $G - e$ is disconnected. An edge e in a disconnected graph G is called a *bridge* if it is a bridge in one of the components of G. When needed, we are able to select a vertex in a tree T, and call it the *root* of T. When we do this, we then refer to the the T as a *rooted tree* and these trees are often redrawn so that the root r is at the top of the graph and the remaining vertices are below the root. For convenience we will denote a rooted tree T with root r as (T, r). If the tree T that we are dealing with is a path, and we select an r such that at most one other vertex is incident to r, then we refer to (T, r) as a *rooted path.* Also, in a tree, any vertex v that has three or more edges incident to it is referred to as a *junction vertex.* Since T is a tree, then T has some vertices which are end vertices. If v is a vertex on the longest path which is also a end vertex, then we call v a /emphperipheral vertex of G. We will denote the set of peripheral vertices by $\Pi(G)$. Any G_i which is itself a tree will also be referred to as a *vertex deleted subtree*. It should also be noted that if G_i is a vertex deleted subtree, then the vertex v_i would have been an end vertex of G.

If we take a tree T and remove all the end vertices, then we will obtain a subtree by repeated application of the previous definition. If we continue this process, eventually the subtree will either be a single vertex or a pair of vertices joined by a single edge. If the result is a single vertex, then we will refer to the tree T as a *central tree* and the remaining vertex as the *center* of T. If the result is a pair of vertices then we will refer to the tree T as a *bicentral tree* and the remaining vertex as the *bicenter*. If T is a central tree, then a *branch* of T, (B, c), is a rooted tree such that the center of T, c, is the root, only one edge incident to c is included, and the collection of vertices, u, where the is a path from c to u that uses that edge. If T is a bicentral tree, then a *branch* of T is one of the components of the subgraph obtained by deleting the single edge which connects the bicenter of T. It follows from this that if T is a bicentral tree, then T has exactly two branches and if T is a central tree, then T has $deg(c)$ branches.

A branch of T is a *peripheral branch* if it contains a peripheral vertex of T. It also follows that every tree will have at least two peripheral branches, in a bicentral tree, both branches are peripheral branches. However, it is possible that a tree T will have more than 2 peripheral branches. This happens when there are more than two vertices in $\Pi(G)$. The number of peripheral branches cannot exceed the number of vertices in $\Pi(G)$.

A pair graphs G and H are called *hypomorphic* if there exists a bijection $\sigma : V(G) \to V(H)$ such that $G - v \cong H - \sigma(v) \forall v \in V(G)$. We refer to the function σ as a *hypomorphism* of G onto H.

Fact 1

A graph G is reconstructable if it has a vertex z such that every z-reconstruction of G is isomorphic to G.

Theorem 6

1) If G and H are hypomorphic, then $|V(G)| = |V(H)|$

2) If G and H are hypomorphic and both have three or more vertices, then $|E(G)| = |E(H)|$.

Proof:

1)

Since G and H are hypomorphic, then there exists a hypomorphism, σ, from G onto H. Then it follows that any hypomorphism of G onto H is a one to one, onto function from $V(G) to V(H)$. Then it follows from the properties of bijections that $|V(G)| = |V(H)|$.

2)

Let σ be a hypomorphism from G onto H. Then because each edge of G is in $|V(G)|-2$ of the G_i and each edge of H is in $|V(H)|-2$ of the H_i and also $G-v \cong H-\sigma(v) \forall v \in V(G)$, it follows that

$|E(G)|(|V(G)|-2) = \sum |E(G-v)|, \forall v \in V(G)$
$= \sum |E(H-\sigma(v))|, \forall v \in V(G)$
$= \sum |E(H-w)|, \forall w \in V(H)$
$= |E(H)|(|V(H-2)|$

Then it follows that $|E(H)| = |E(G)|$ because $|V(G)| \geq 3$ and $|V(H)| \geq 3$.//

Theorem 7

If two graphs of order greater than or equal to three are hypomorphic then they are either both connected or both disconnected.

Proof:

Let G and H be two graphs of order $n \geq 3$. Assume that σ is a hypomorphism from G onto H. We know from Corollary 1 that if G and H are connected then they both contain at least two vertices that are not cut vertices and that they both have at least two connected vertex deleted subgraphs. If G and H are disconnected then it follows that they must have at most one connected vertex deleted subgraph. Then because G and H have the same number of connected and disconnected vertex deleted subgraphs, it follows that if G is connected then so is H and if G is disconnected then H must be also.//

Lemma 1

Any graph hypomorphic to a tree with at least three vertices, is also a tree with at least three vertices.

Proof:

Assume that G is a graph which is hypomorphic to a tree T which has at least 3 vertices. Then by Theorem 6, the order of G is the same as the order of T. Let the order of G be n. Since T is a tree, then T has $n-1$ edges. Also, because there is a hypomorphism between G and T, Theorem 6 says that the number of edges in G and T is the same. Therefore, G has $n-1$ edges. It also follows from Theorem 7 that G is connected. Then by the definition of a tree, G is a tree.//

Lemma 2

Every tree T with at least 3 vertices and at most one junction vertex is reconstructable.

Proof:

Let T be a tree with at least 3 vertices and at most one junction vertex. Let G be a graph which is hypomorphic to T. Then, by the definition of hypomorphic, there exists a hypomorphism σ of T onto G. Then it follows from lemma 1 that G is a tree.

Case 1

Assume T has no junction vertices. Then it follows that also G has no junction vertices. Then by the definition of a junction vertex, it follows that T

and G are both paths. Therefore, T is isomorphic to G and it then follows that G is the reconstruction of T.

Case 2

Assume T has one junction vertex. Assume that v is the junction vertex of T and by the definition of a hypomorphism, $\sigma\ (v)$ is the only junction vertex of G. Then from this information it follows that G is isomorphic to T and more so that G is the reconstruction of $T._{//}$

We will refer to a *v-reconstruction* of a graph G as a graph H where $V(G) = V(H)$, $G - v = H - v$ and G is hypomorphic to H. Since in a v-reconstruction of a graph G, any $v \in V(G)$ is also a $v \in V(H)$ then if we are referring to a vertex that is in one of these sets while looking at a v-reconstruction of H, in consideration of the degree of the vertex $v, deg_G(v)$ will be the degree of v in the graph G and $deg_H(v)$ will be the degree of v in H. For our discussion, we will use the convention that the set of vertices, $N_G(v)$ of vertices in G which are adjacent to v will be called the *neighborhood* of v. We will also let Y_α denote a tree which has exactly one junction vertex and exactly three end vertices, in which the junction vertex is adjacent to two and only two of the end vertices and for any positive integer α the distance between the junction vertex and the non-adjacent end vertex is α.

Looking at the set of vertices of G, $V(G)$, we will refer to $V_1(G)$ as the set of the end vertices of G. It also follows that $V_1(G)$ is a subset of $V(G)$. Also, a vertex of a graph G is *bad* is G has a vertex of degree $n = deg(v) - 1$. Assuming that G and Q are graphs, we will let $s_Q(G)$ denote the number of subgraphs of G which are isomorphic to Q. More specifically we will let $s_Q(G, v)$ be the subgraphs of G which include the vertex v that are isomorphic to Q. We will also let $d_T(u, v)$ denote the distance in tree T between vertices u and v. Assuming that U is a nonempty subset of $V(T)$, then we will let $d_T(u, U)$ denote the minimum $d_T(u, v) \forall v \in U$. If T is a tree with three or more vertices and (R, r) is a rooted tree, we will let $b_{(R,r)}(T)$ denot the number of branches of T that are isomorphic to (R, r).

Lemma 3, Kelly's Lemma

If G and H are hypomorphic graphs, Q a graph such that $|V(Q)| < |V(G)|$, then $s_Q(G) = s_Q(H)$.

Proof:

Since G and H are hypomorphic, then there exists a hypomorphism, σ, of G onto H. Since each subgraph of G which is isomorphic to Q is contained in $|V(G)| - |V(Q)|$ of our G_i and each subgraph of H which is isomorphic to Q is contained in $|V(H)| - |V(Q)|$ of our H_i and also $G - v \cong H - \sigma(v) \forall v \in V(G)$, then it follows that

$$s_Q(G)(|V(G)| - |V(Q)|) = \sum s_Q(G - v), \forall v \in V(G)$$
$$= \sum s_Q(H - \sigma(v)), \forall v \in V(G)$$
$$= \sum s_Q(H - w), \forall w \in V(H)$$
$$= s_Q(H)(|V(H)| - |V(Q)|).$$

From this we can conclude that $s_Q(G) = s_Q(H)$ since $|V(Q)| < |V(H)|$ and $|V(Q)| < |V(G)|$ by theorem 6.$_{//}$

Lemma 4

If v is a vertex of a graph G and if H is a v-reconstruction of G and if Q is a graph such that $|V(Q)| < |V(G)|$, then $s_Q(G, v) = s_Q(H, v)$.

Proof:

Since $G - v = H - v$, then by Lemma 3 we get that

$s_Q(G, v) = s_Q(G) - s_Q(G - v)$

$s_Q(H) - s_Q(H - v)$

$s_Q(H, v).{}_{//}$

Lemma 5

Let z be a vertex of a graph G with at least three vertices. Let H be a v-reconstruction of G. Then for some non-negative integer m, there exist m distinct bad neighbors, $v_1, \cdots, v_m$, of z in G and m distinct vertices, $u_1, \cdots, u_m$ of $V(G) - N_G(z)$ such that

$deg_G(u - i) = deg_G(v_i) - 1, \forall i = 1, \cdots, m$

and

$N_H(z) = (N_G(z) - \{v_1, \cdots, v_m\}) \bigcup \{u_1, \cdots, u_m\}$

Proof:

Since H is hypomorphic to G and $H - z = G - z$, then it follows that $deg_G(z) = deg_H(z)$. For all nonnegative integers k the number of vertices in $N_H(z) - N_G(z)$ who have degree k in H must be equal to the number of vertices in $N_G(z) - N_H(z)$ who have degree k. So we write

$N_G(z) - N_H(z) = \{v_1, \cdots, v_m\}$

$N_H(z) - N_G(z) = \{u_1, \cdots, u_m\}$

with

$deg_G(v_i) = deg_H(w_i) = deg_G(w_i) + 1, i = 1, \cdots, m.$

It follows from this that $v_1, \cdots, v_m$ are bad in G and $u_1, \cdots, u_m$ are the vertices of G such that $deg_G(u_i) = deg_G(v_i) - 1$.

We still must show that $N_H(z) = (N_G(z) - \{v_1, \cdots, v_m\}) \bigcup \{u_1, \cdots, u_m\}$. However, using substitution this result easily follows from the above information.$_{//}$

Lemma 6

Let T be a tree and either

(i) T has exactly two branches, and at least one of them is a rooted path or

(ii) T is a central tree and all of its peripheral branches are rooted paths.

Then T is reconstructable.

Proof:

From Lemma 2 we can assume that G has more than one junction vertex. Assume that the diameter of T is a. By the assumptions of the lemma, T has a peripheral branch that is a rooted path. Let v be the peripheral vertex on that branch. Let w be the neighbor of v in T. Assume that S is a v-reconstruction of T. Then by Fact 1, it remains to prove that $T \cong S$.

Since T has at least two junction vertices, we can then conclude that $deg(w) = 2$. Then, by Lemma 5, either $N_S(v) = \{w\}$ or $N_S(v) = \{z\}$ for some $z \in V_1(T) - \{v\}$.

If $N_S(v) = \{w\}$ it is obvious that $S \cong T$.

Assume that $N_S(v) = \{z\}$ for some $z \in V_1(T) - \{v\}$. In this case, it is obvious that the set of junction vertices of T, $J(T)$ is the same as the set of junction vertices of S, $J(S)$. It also follows thats S is a tree. We will put the set J equal to $J(T)$. Then by Lemma 4, the smallest positive integer α such that v is in a subtree of T that is isomorphic to Y_α is the same as the smallest α such that v is in a subtree of S that is isomorphic to Y_α. In other words,

$d_T(v, J) = d_S(v, J) = d_T(z, J) + 1$

It also follows from the way that we defined v and the fact that the diameter of T is larger than $d_T(v, z)$ that $d_T(v, J) \geq \frac{1}{2}a$. Then it follows, with a little work which will be omitted at this time, that the vz-path in T includes exactly one element in J, call it u. Then it follows that $d_T(u, v) = d_T(z, u) + 1$. Therefore, $S \cong T._{//}$

Trees are reconstructable.

Proof:

Let T and U be hypomorphic trees of order greater than or equal to three. Then by Lemma 1, we need to prove that $S \cong T$.

If Lemma 6 applies to either U or T, then we know that because they are hypomorphic and one of them is reconstructable that $S \cong T$. Assume both U and T fail at least one of the hypotheses of Lemma 6.

The by Lemma 3, $s_Q(U) = s_Q(T)$. Let l be the path of longest length in T. Then it follows that U must contain a path of length l and that it must be the longest path in U. This means that U and T have the same diameter, let $d = diam(U) = diam(T)$. Define a set $D(T)$ as the set of all vertex deleted subtrees of T which have diameter d. Then $D(U)$ represents the same thing for U. Since U and T are hypomorphic, then every tree in $D(T)$ is isomorphic to a tree in $D(U)$ and the same for trees in $D(U)$ being isomorphic to trees in $D(T)$. Then it follows that if $b_{(R,r)}$ is the order of the largest branch in T, then $b_{(R,r)}$ is the order of the largest branch in U also. Pick (R, r) a rooted tree of order $b_{(R,r)}$ where at least one subtree in $D(U)$ has a branch isomorphic to (R, r).

We need to show that (R, r) is not a rooted path, however we will omit this work at this time and assume that (R, r) is not a rooted path. Since (R, r) is not a rooted path, we can choose a vertex, $v \in v_1(R) - \{r\}$ so that $d_R(r, w) = \frac{1}{2}a$ for some vertex $w \in V_1(M) - \{v\}$. Put $L = M - v$. Now pick an element $S - z$ of $D(S)$ such that

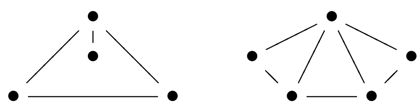
1) $b_{(R,r)}(S - z) \leq b_{(R,r)}(S - u), \forall S - u \in D(S)$

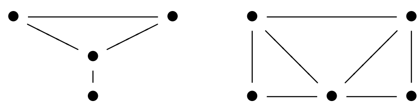
2) $b_{(L,r)}(S - z) \geq b_{(L,r)}(S - u), \forall S - u \in D(S) such that b_{(R,r)}(S - z) = b_{(R,r)}(S - u)$

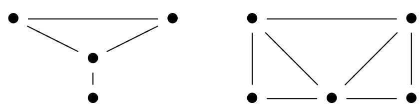
Let $T - x \in D(T)$ such that $T - x \cong S - u$.

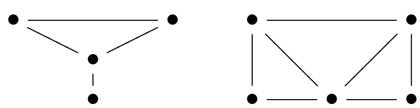
Since S does not satisfy Lemma 6, we are assured that each branch of S is a member of $D(S)$. Also becase T does the same, we know that each branch of T is a member of $D(T)$. Then following the same idea that we used in the proof that disconnected graphs are reconstrutable, we see that S is isomorphic to a tree obtained from the graph $S - u$ by replacing one of the branches that is isomorphic to (L, r) with one that is isomorphic to (R, r). If we do a similar style replacement with T. Then combining this with he fact that $S - u \cong T - x$ we conclude that $S \cong T$.$_{//}$

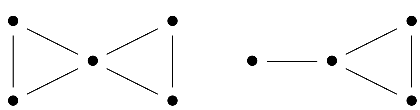
Example Reconstruction

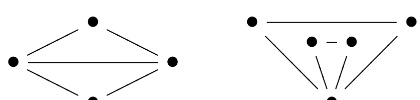
$G_1 =$ 

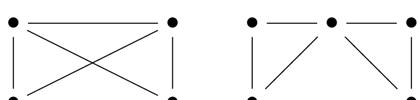
$G_2 =$ 

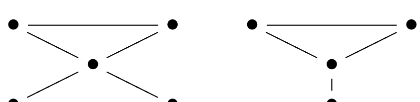
$G_3 =$ 

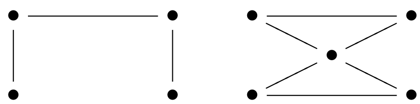
$G_4 =$ 

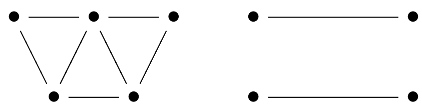
$G_5 =$ 

$G_6 =$ 

$G_7 =$ 

$G_8 =$ 

$G_9 =$ 

$G_{10} =$ 

First we can easily see that n for our original G is 10.

Using the formulas that we have previously introduced, we can compute p. We know that

$$q = \sum \frac{\mathrm{q_i}}{\mathrm{n-2}}$$

We can easily see that

$q_1 = 11$
$q_2 = 11$
$q_3 = 11$
$q_4 = 11$
$q_5 = 10$
$q_6 = 11$
$q_7 = 11$
$q_8 = 10$
$q_9 = 9$
$q_{10} = 9.$

So from that we get

$$q = \frac{11+11+11+11+10+11+11+10+9+9}{10-2}$$

$$q = \frac{104}{8} = 13$$

Another formula that we have previously introduced will give us the degree of each v_i.

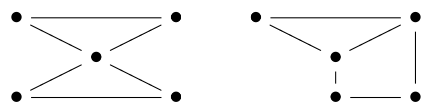
$$degv_i = q - q_i$$

So

$degv_1 = q - q_1 = 13 - 11 = 2$
$degv_2 = q - q_2 = 13 - 11 = 2$
$degv_3 = q - q_3 = 13 - 11 = 2$
$degv_4 = q - q_4 = 13 - 11 = 2$
$degv_5 = q - q_5 = 13 - 10 = 3$
$degv_6 = q - q_6 = 13 - 11 = 2$
$degv_7 = q - q_7 = 13 - 11 = 2$
$degv_8 = q - q_8 = 13 - 10 = 3$
$degv_9 = q - q_9 = 13 - 9 = 4$
$degv_{10} = q - q_{10} = 13 - 9 = 4.$

We know that the original graph, G, is disconnected because each of the $G - v_i$ subgraphs are disconnected.

Since the Reconstruction Conjecture has not been proven true, we do not know that we can uniquely reconstruct G. However, one possible reconstruction of G is

$G =$ 

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