Perturbational evaluation of bend mode phase shifts for the tuning of cylindrical microresonators

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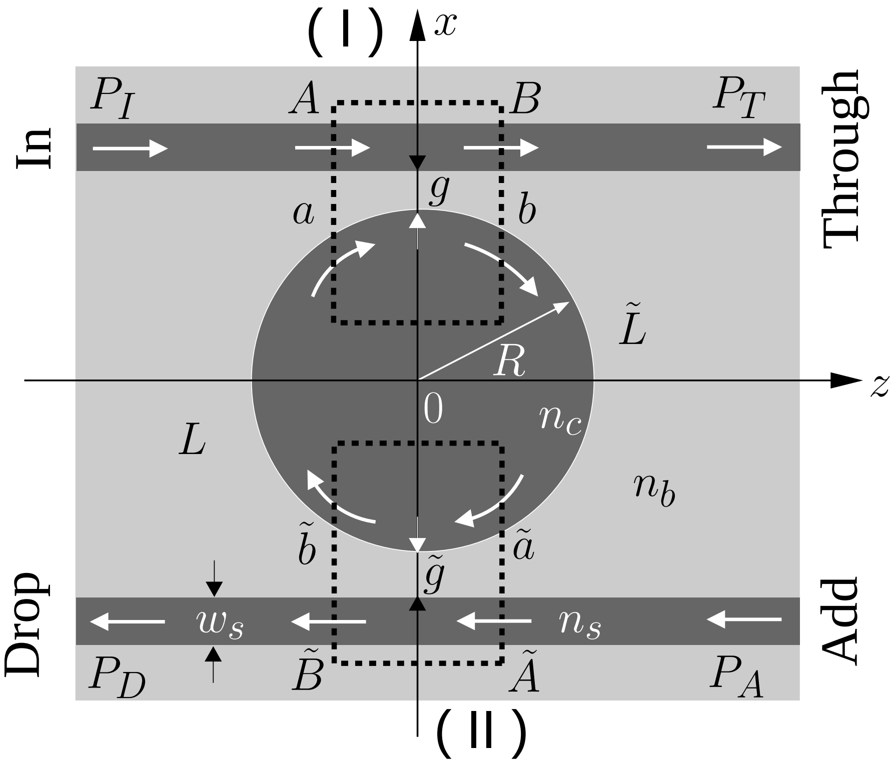
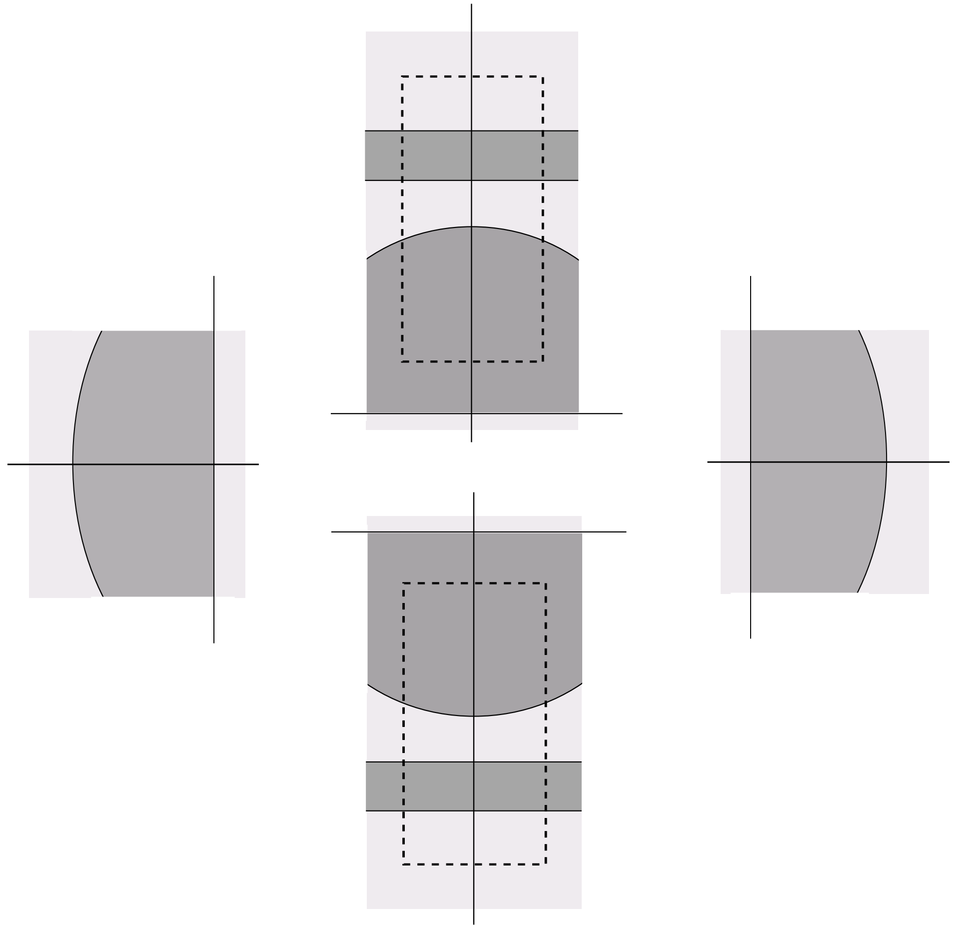
Schematic representation of 2D microresonator





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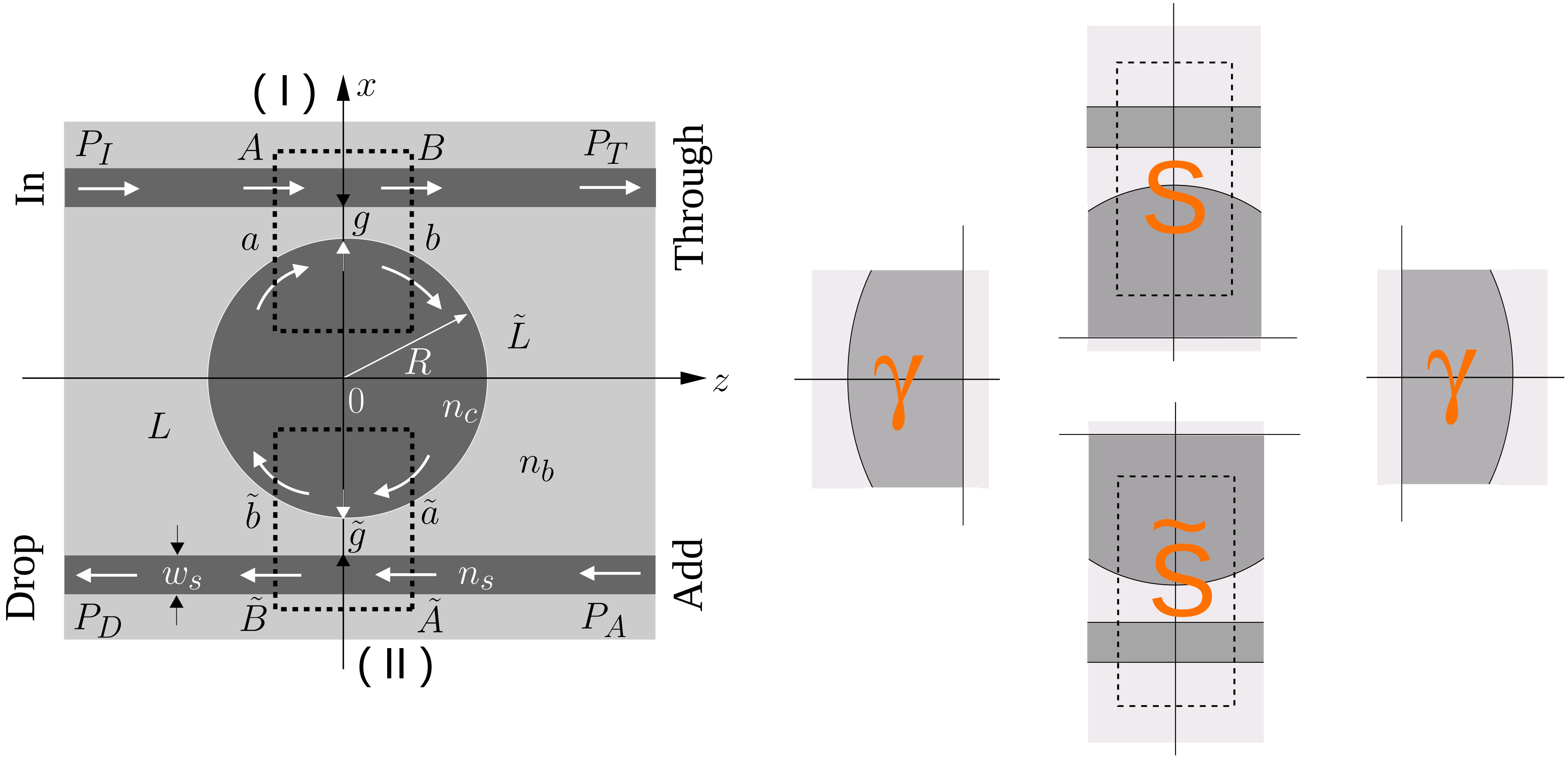
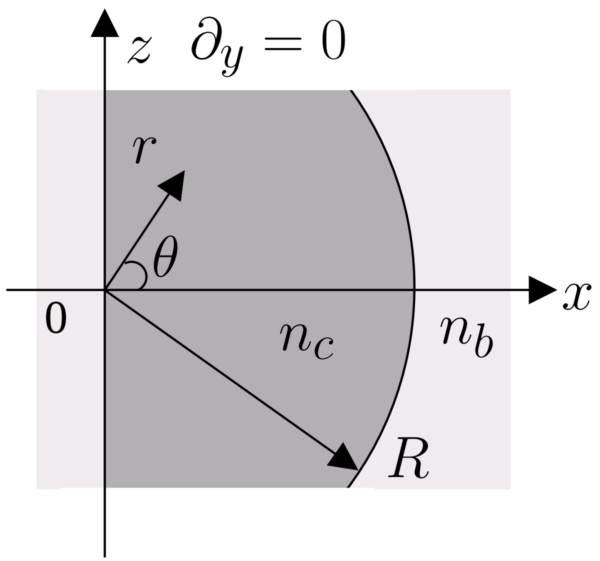


Bent Waveguide



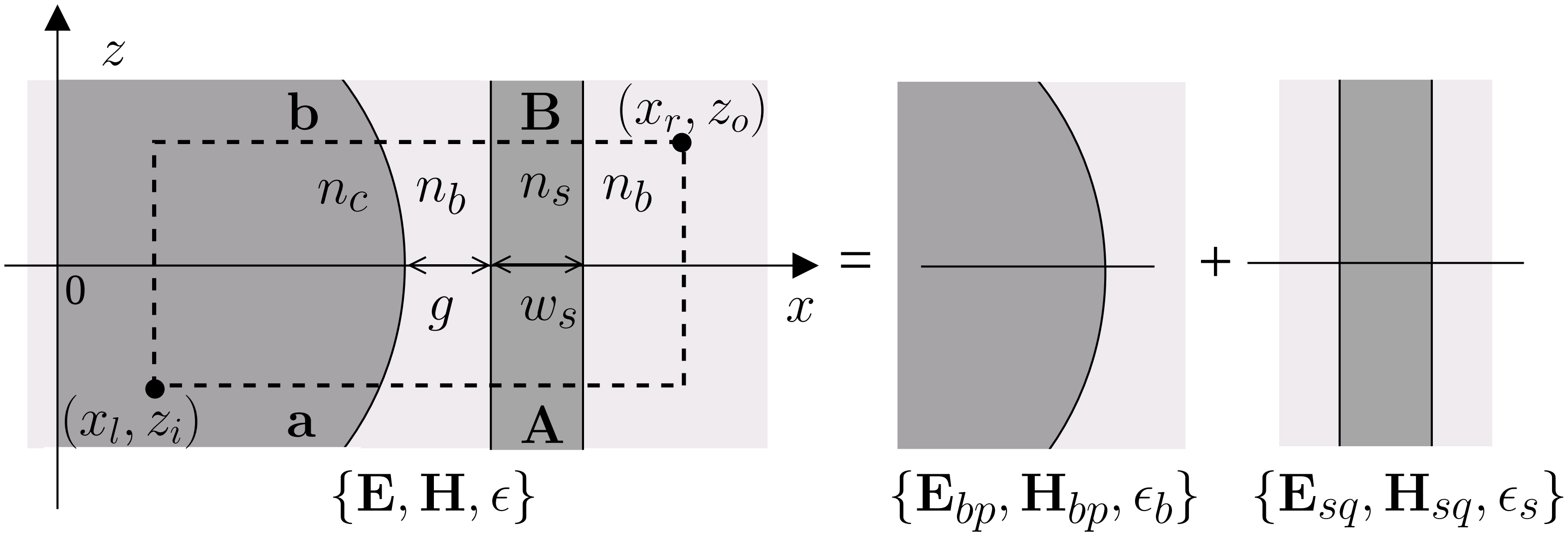
- Field ansatz:

$$\mathbf{E} = (\tilde{E}_r(r), \tilde{E}_y(r), \tilde{E}_\theta(r))\mathrm{e}^{\mathrm{i}(\omega t - \gamma R\theta)}$$

$$\mathbf{H} = (\tilde{H}_r(r), \tilde{H}_y(r), \tilde{H}_\theta(r))\mathrm{e}^{\mathrm{i}(\omega t - \gamma R\theta)}$$

- $\omega \in \mathbb{R}$
- Propagation constant $\gamma = \beta - i\alpha \ \in \mathbb{C}$
- Basic components $\tilde{E}_y$ (TE) and $\tilde{H}_y$ (TM)
- *Bessel equation* for $\tilde{E}_y$ and $\tilde{H}_y$ of order γR
- Piecewise field ansatz + Interface conditions $\Rightarrow$ Dispersion equation
- Dispersion equation + Complex order Bessel functions $\Rightarrow$ Analytical Bend Modes

Bent-straight waveguide coupler



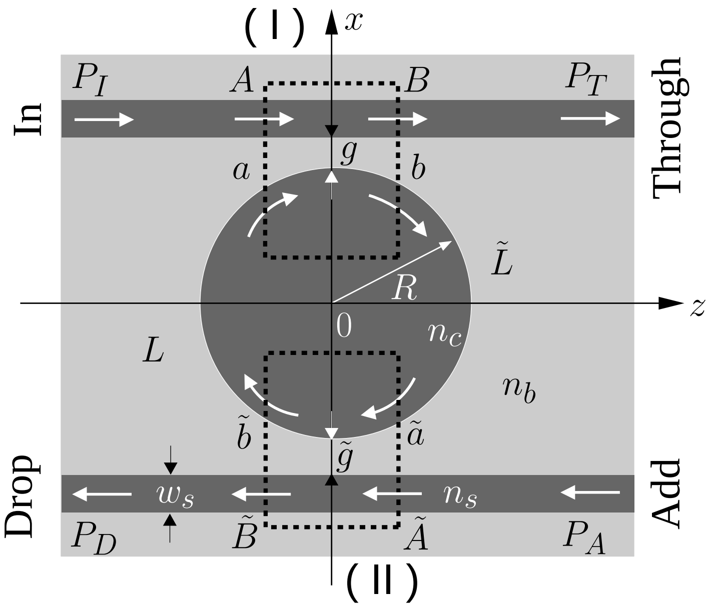
- Coupled Mode ansatz: C_{bp}, C_{sq} unknown amplitudes

$$\begin{pmatrix} \mathbf{E}(x,z) \\ \mathbf{H}(x,z) \end{pmatrix} = \sum_{p=1}^{N_b} C_{bp}(z) \begin{pmatrix} \mathbf{E}_{bp}(x,z) \\ \mathbf{H}_{bp}(x,z) \end{pmatrix} + \sum_{q=1}^{N_s} C_{sq}(z) \begin{pmatrix} \mathbf{E}_{sq}(x,z) \\ \mathbf{H}_{sq}(x,z) \end{pmatrix}$$

- Lorentz reciprocity theorem / variational method $\mapsto$ $\mathsf{M}(z) \cdot d_z \mathbf{C}(z) = \mathsf{F}(z) \cdot \mathbf{C}(z)$
- Solve for $\mathbf{C}$ + projection on straight waveguide modes $\mapsto$ scattering matrix (S)

Microresonator response

$$\begin{pmatrix} \mathbf{b} \\ \mathbf{B} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathsf{S}_{bb} & \mathsf{S}_{bs} \\ \mathsf{S}_{sb} & \mathsf{S}_{ss} \end{pmatrix}}_{\mathsf{S}} \begin{pmatrix} \mathbf{a} \\ \mathbf{A} \end{pmatrix}, \quad \begin{pmatrix} \tilde{\mathbf{b}} \\ \tilde{\mathbf{B}} \end{pmatrix} = \underbrace{\begin{pmatrix} \tilde{\mathsf{S}}_{bb} & \tilde{\mathsf{S}}_{bs} \\ \tilde{\mathsf{S}}_{sb} & \tilde{\mathsf{S}}_{ss} \end{pmatrix}}_{\tilde{\mathsf{S}}} \begin{pmatrix} \tilde{\mathbf{a}} \\ \tilde{\mathbf{A}} \end{pmatrix}, \quad \underbrace{\mathbf{a} = \tilde{\mathsf{G}}\,\tilde{\mathbf{b}}, \quad \tilde{\mathbf{a}} = \mathsf{G}\,\mathbf{b}}_{\mathsf{G},\tilde{\mathsf{G}}\ \mathrm{diag}(\mathrm{e}^{-\mathrm{i}\gamma_j R\theta})}$$

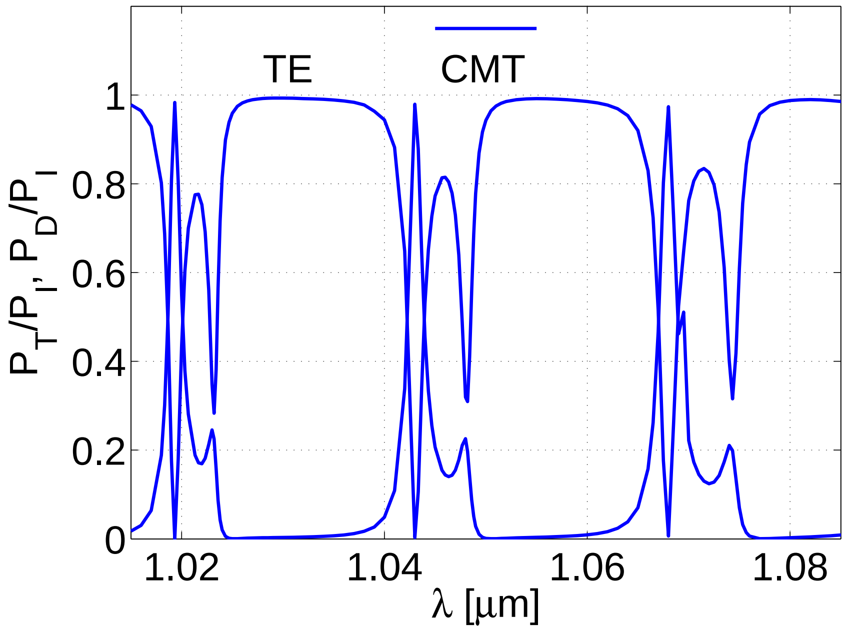


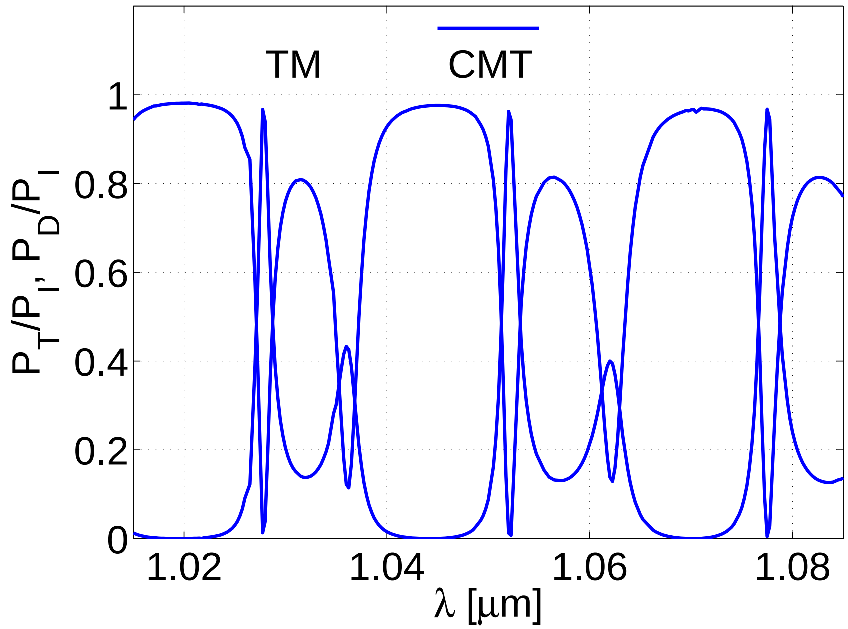
- Redefine S, $\tilde{\mathsf{S}}$ at a point $\mapsto L + \tilde{L} = 2\pi R$
- Given $A = \sqrt{P_I}$, $\tilde{A} = 0$.
  Solve for $P_T = |B|^2$ and $P_D = |\tilde{B}|^2$.

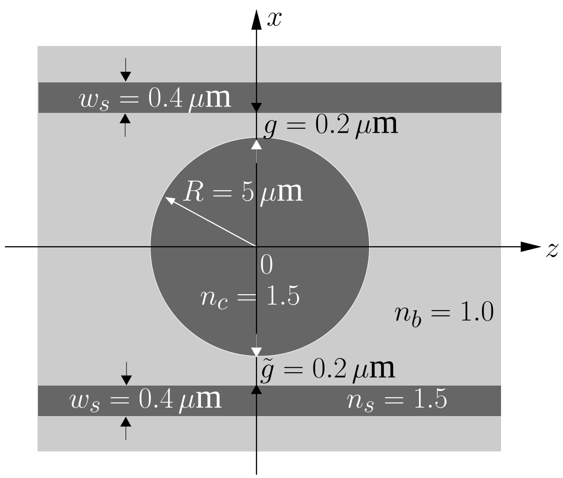
$$\begin{aligned} \mathbf{B} &= (\mathsf{S}_{sb}\mathsf{G}\tilde{\mathsf{S}}_{bb}\tilde{\mathsf{G}}\Omega^{-1}\mathsf{S}_{bs} + \mathsf{S}_{ss})\mathbf{A} \\ \tilde{\mathbf{B}} &= (\tilde{\mathsf{S}}_{sb}\tilde{\mathsf{G}}\Omega^{-1}\mathsf{S}_{bs})\mathbf{A} \\ \Omega &= \mathsf{I} - \mathsf{S}_{bb}\mathsf{G}\tilde{\mathsf{S}}_{bb}\tilde{\mathsf{G}} \end{aligned}$$

- Solutions for a series of wavelengths $\mapsto$ spectral response

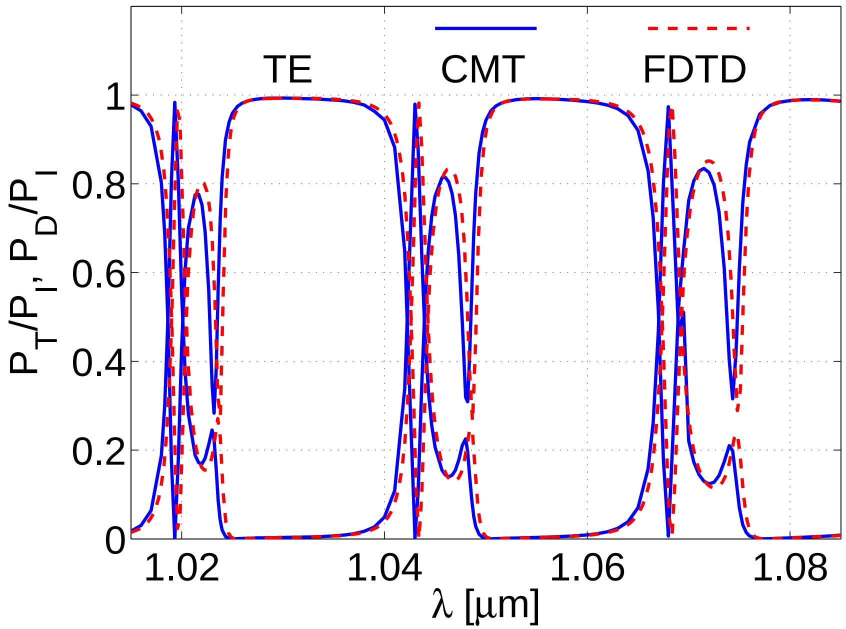
Spectral response

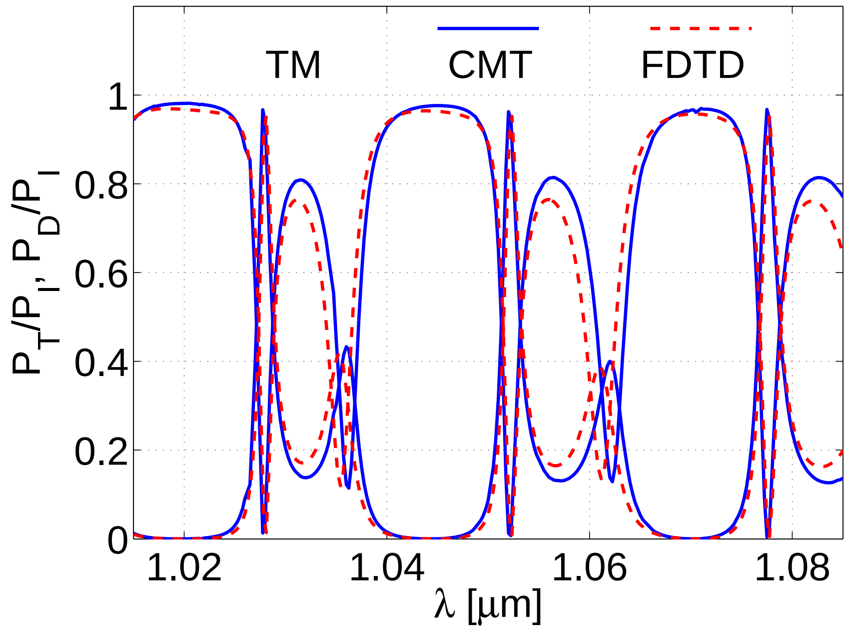


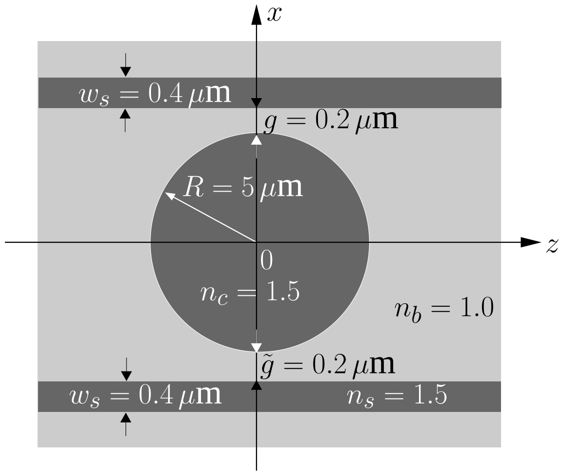




Spectral response

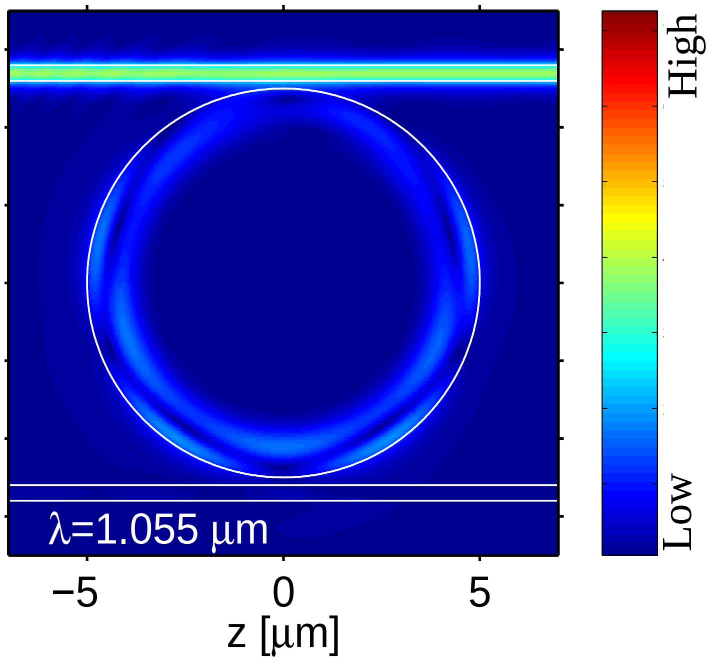


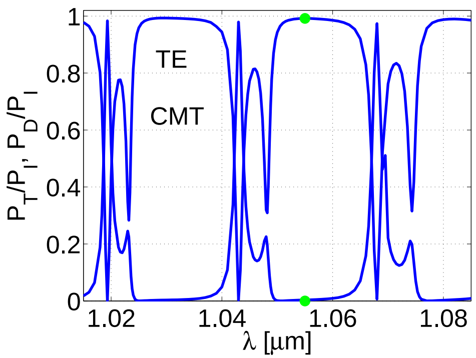




Microresonator field profiles

- TE $|E_y|$: Coupled mode theory microresonator simulations with 3 cavity modes

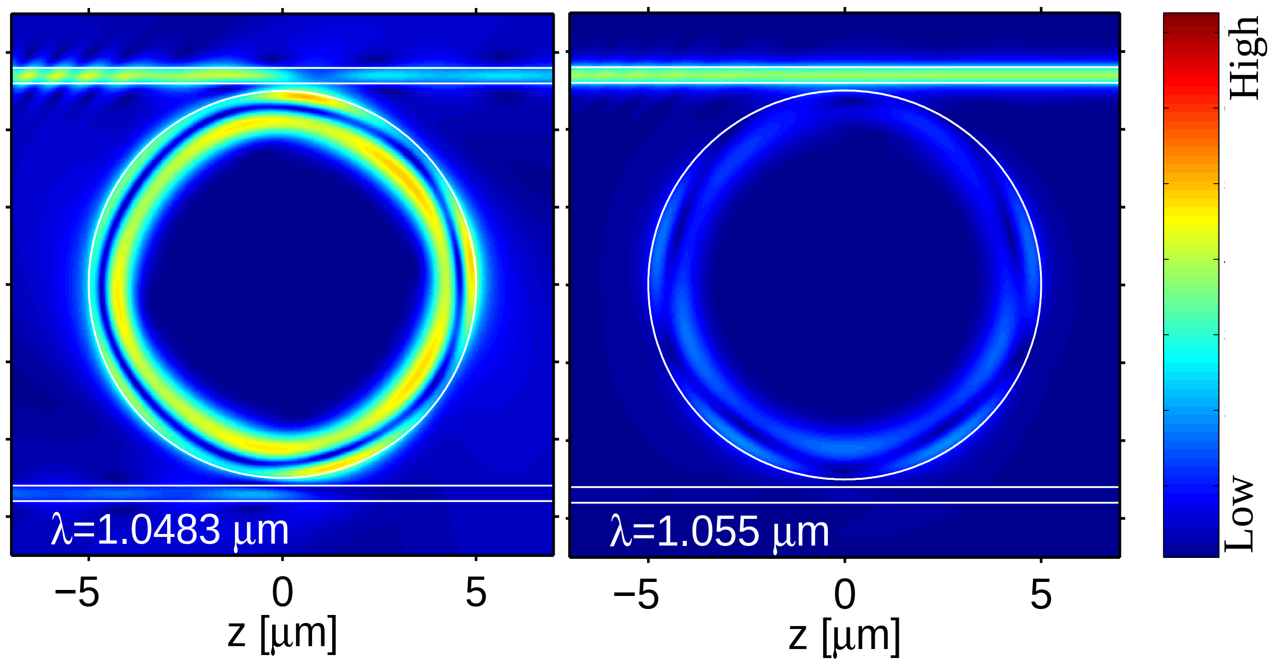


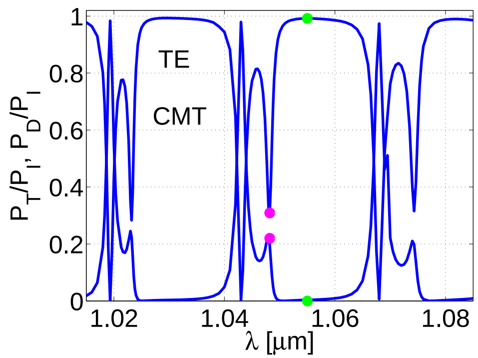


- $\lambda = 1.055\,\mu$m $\mapsto$ off resonance

Microresonator field profiles

- TE $|E_y|$: Coupled mode theory microresonator simulations with 3 cavity modes

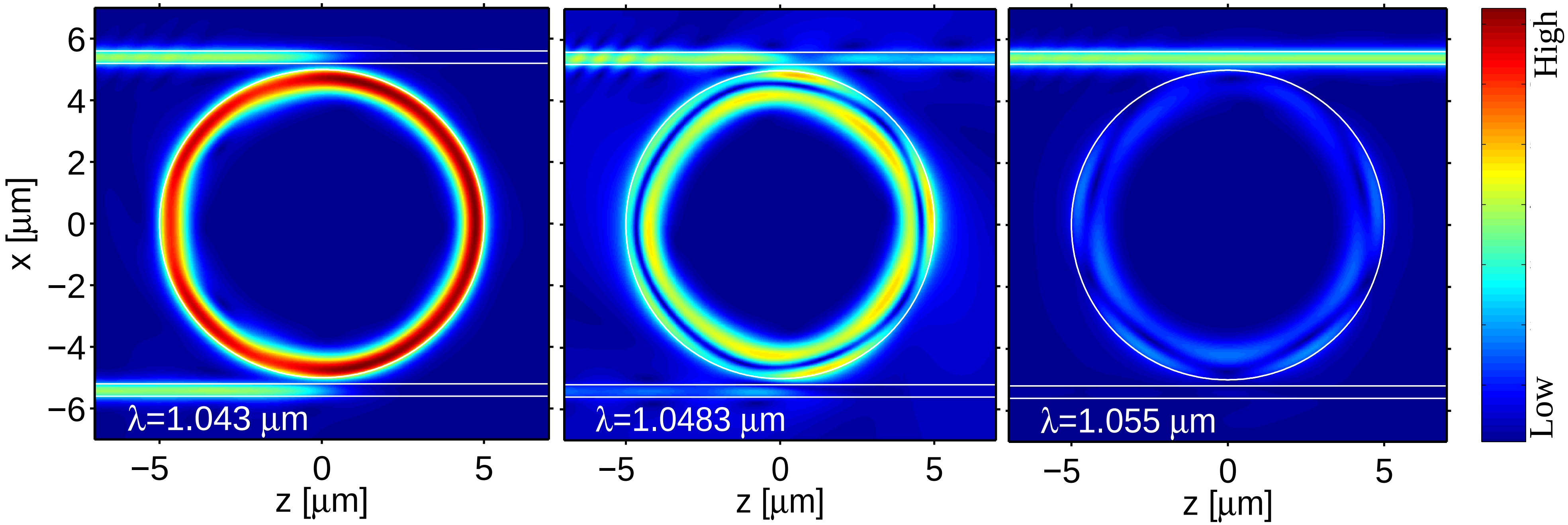


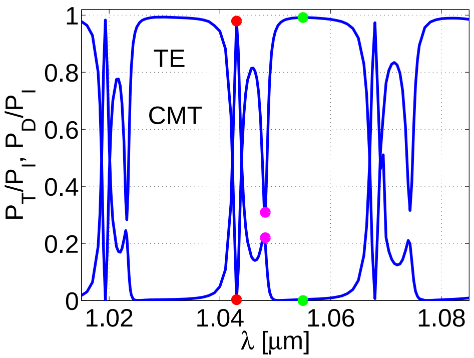


- $\lambda = 1.0483\,\mu$m $\mapsto$ Resonance corresponding to first order mode
- $\lambda = 1.055\,\mu$m $\mapsto$ off resonance

Microresonator field profiles

- TE $|E_y|$: Coupled mode theory microresonator simulations with 3 cavity modes





- $\lambda = 1.043\,\mu$m $\mapsto$ Resonance corresponding to fundamental mode
- $\lambda = 1.0483\,\mu$m $\mapsto$ Resonance corresponding to first order mode
- $\lambda = 1.055\,\mu$m $\mapsto$ off resonance

Tuning

- Over a small wavelength interval, the scattering matrix is assumed to be constant. Phase relations of the cavity modes $\leftrightarrow$ resonances

- At resonance $2\pi R\beta_m + \phi = 2\pi m$
  ϕ: Phase change due to straight waveguide coupling

- Tuning: p perturbation (tuning) parameter

$$\beta(p, \tilde{\lambda}_m) \approx \beta(0, \lambda_m) + p \left.\frac{\partial \beta}{\partial p}\right|_{0,\lambda_m} + (\tilde{\lambda}_m - \lambda_m) \left.\frac{\partial \beta}{\partial \lambda}\right|_{0,\lambda_m} \stackrel{!}{=} \beta_m$$

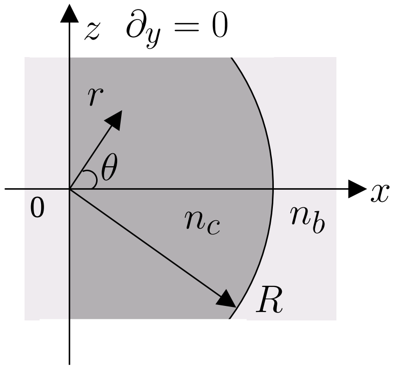
$$\text{Shift due to tuning : } \Delta\lambda = p \left.\left(\frac{\partial \beta}{\partial p}\right)\left(\frac{\partial \beta}{\partial \lambda}\right)^{-1}\right|_{0,\lambda_m}$$

- Homogeneity arguments: $\dfrac{\partial \beta}{\partial \lambda} \approx -\dfrac{\beta}{\lambda}$

- For cavity core permittivity perturbation $p = \delta\epsilon_c$ $\mapsto$ How to find $\dfrac{\partial \beta}{\partial \epsilon_c}$?

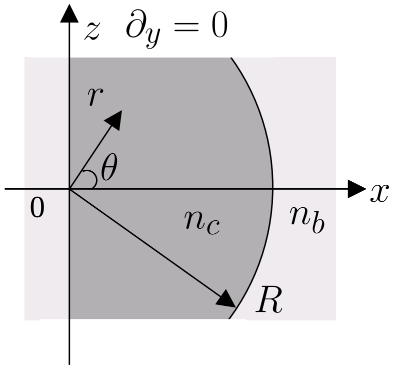
Permittivity perturbation

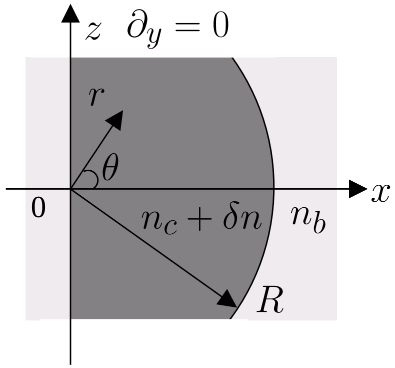
$$\underbrace{\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{pmatrix} (r)\mathsf{e}^{-\mathsf{i}\gamma R\theta}}_{\text{Unperturbed } n_c = \sqrt{\epsilon_c}}$$



Permittivity perturbation

$$\underbrace{\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}(r)\mathrm{e}^{-\mathrm{i}\gamma R\theta}}_{\text{Unperturbed } n_c=\sqrt{\epsilon_c}}$$

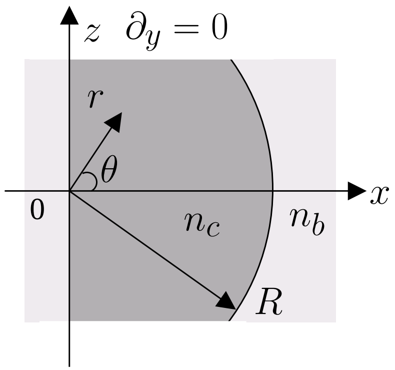


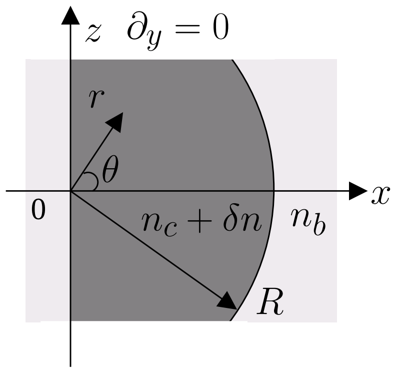


$$\underbrace{\begin{pmatrix}\mathbf{E}_p\\ \mathbf{H}_p\end{pmatrix} = C(\theta)\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}}_{\text{Perturbed } n_c+\delta n=\sqrt{\epsilon_p}}$$

Permittivity perturbation

$$\underbrace{\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}(r)\mathrm{e}^{-\mathrm{i}\gamma R\theta}}_{\text{Unperturbed } n_c=\sqrt{\epsilon_c}}$$





$$\underbrace{\begin{pmatrix}\mathbf{E}_p\\ \mathbf{H}_p\end{pmatrix} = C(\theta)\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}}_{\text{Perturbed } n_c+\delta n=\sqrt{\epsilon_p}}$$

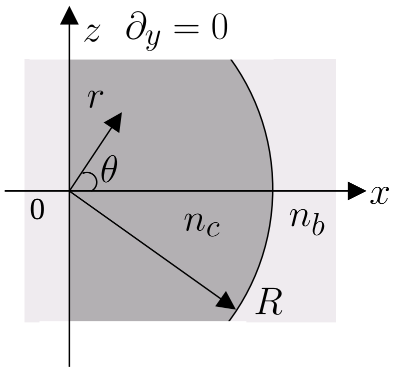
- Lorentz's reciprocity theorem: $(\mathbf{E}_p, \mathbf{H}_p, \epsilon_p)$ and $(\mathbf{E}, \mathbf{H}, \epsilon_c) \mapsto \mathsf{M}\cdot d_\theta C(\theta) = \mathsf{F}\cdot C(\theta)$

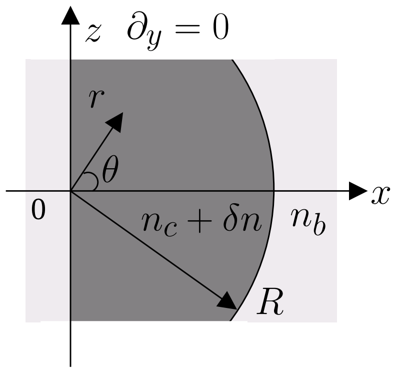
$$\begin{pmatrix}\mathbf{E}_p\\ \mathbf{H}_p\end{pmatrix} = C_0\begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}(r)\mathrm{e}^{-\mathrm{i}(\gamma+\delta\gamma)R\theta} \qquad \delta\gamma = \frac{\omega\epsilon_0}{R}\frac{\int_0^R(\epsilon_p-\epsilon_c)\mathbf{E}\cdot\mathbf{E}^*\ r\ dr}{\int_0^\infty \mathbf{a}_\theta\cdot(\mathbf{E}\times\mathbf{H}^*+\mathbf{E}^*\times\mathbf{H})\ dr}$$

Permittivity perturbation

$$\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix} = \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}(r)\mathrm{e}^{-\mathrm{i}\gamma R\theta}$$

Unperturbed $n_c = \sqrt{\epsilon_c}$





$$\begin{pmatrix}\mathbf{E}_p\\ \mathbf{H}_p\end{pmatrix} = C(\theta)\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}$$

Perturbed $n_c + \delta n = \sqrt{\epsilon_p}$

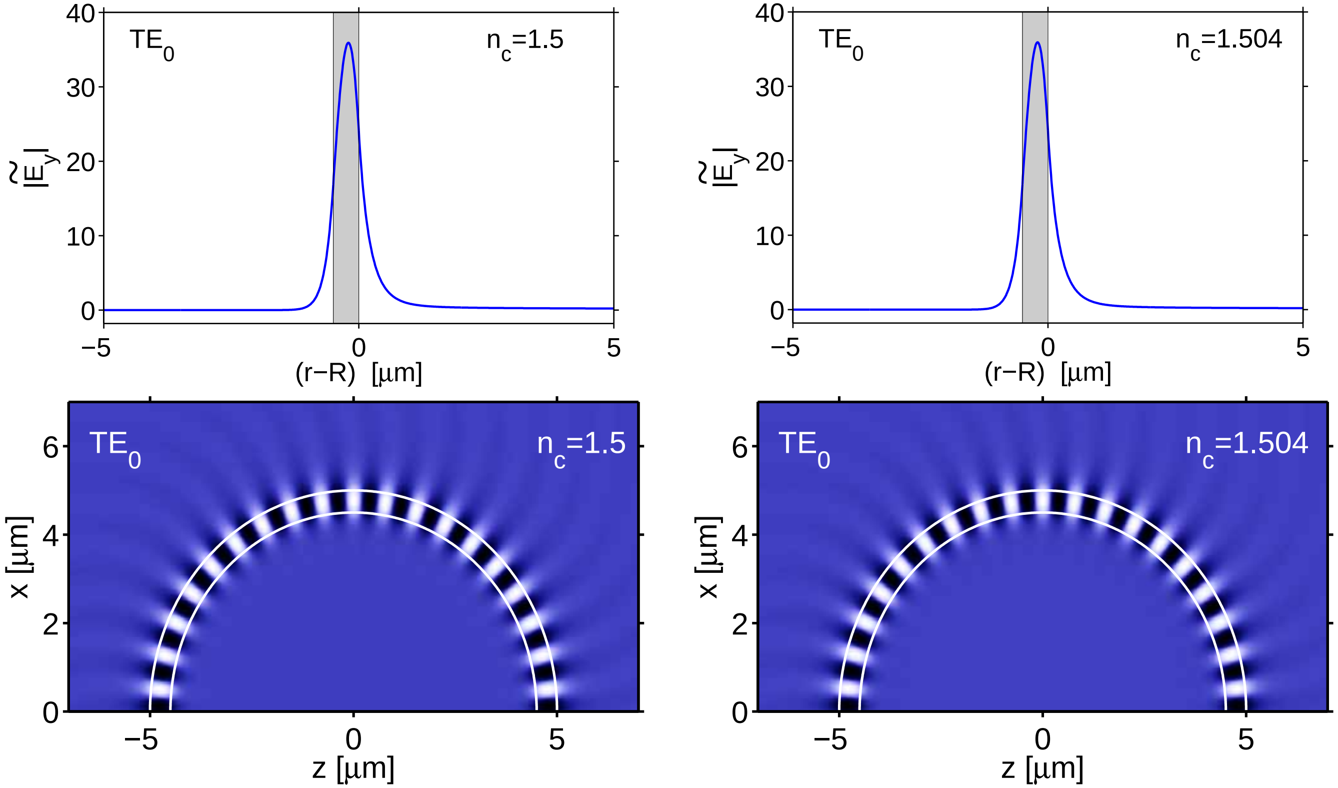
- Lorentz's reciprocity theorem: $(\mathbf{E}_p, \mathbf{H}_p, \epsilon_p)$ and $(\mathbf{E}, \mathbf{H}, \epsilon_c) \mapsto \mathsf{M} \cdot d_\theta C(\theta) = \mathsf{F} \cdot C(\theta)$

$$\begin{pmatrix}\mathbf{E}_p\\ \mathbf{H}_p\end{pmatrix} = C_0 \begin{pmatrix}\tilde{\mathbf{E}}\\ \tilde{\mathbf{H}}\end{pmatrix}(r)\mathrm{e}^{-\mathrm{i}(\gamma+\delta\gamma)R\theta} \qquad \delta\gamma = \frac{\omega\epsilon_0}{R}\frac{\int_0^R (\epsilon_p - \epsilon_c)\mathbf{E}\cdot\mathbf{E}^*\ r\ dr}{\int_0^\infty \mathbf{a}_\theta \cdot (\mathbf{E}\times\mathbf{H}^* + \mathbf{E}^*\times\mathbf{H})\ dr}$$

- $\delta\gamma$ is real $\Longrightarrow$ change in phase constant $(\delta\beta)$

$$\frac{d\beta}{d\epsilon_c} \approx \frac{\delta\beta}{\delta\epsilon_c} = \frac{\omega\epsilon_0}{R}\frac{\int_0^R \tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^*\ r\ dr}{\int_0^\infty \mathbf{a}_\theta \cdot (\tilde{\mathbf{E}}\times\tilde{\mathbf{H}}^* + \tilde{\mathbf{E}}^*\times\tilde{\mathbf{H}})\ dr}$$

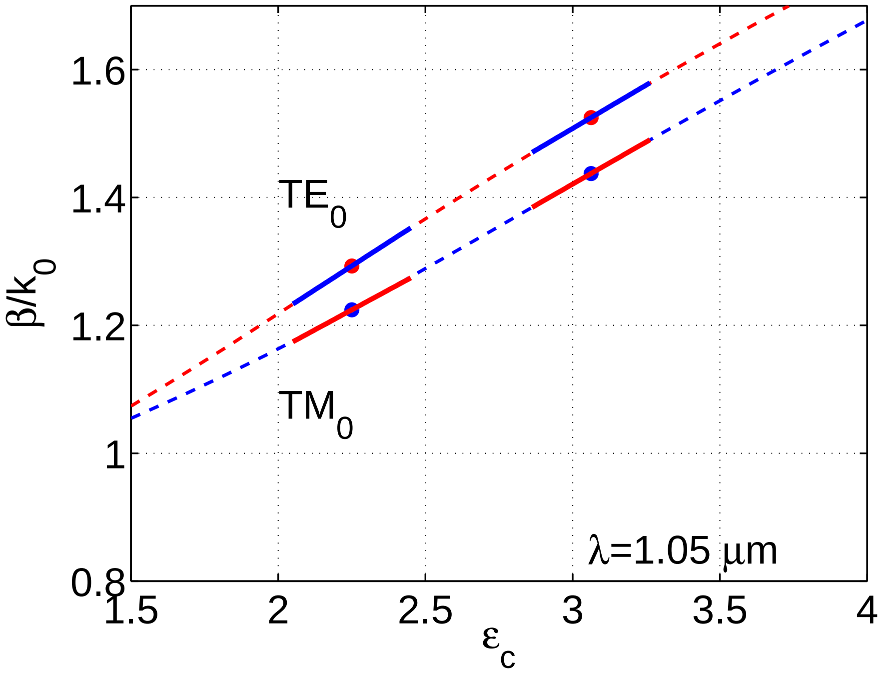
Bend modes



$|\tilde{E}_y|$ *and real physical* E_y *for a bent waveguide* $n_b = 1.0$, *width*$=0.5\,\mu m$, $R = 5\,\mu m$, $\lambda = 1.05\,\mu m$.

	$n_c = 1.5$	$n_c = 1.504$	
	N_{eff} (Direct)	N_{eff} (Direct)	$\Re(N_{\text{eff}})$ (Perturb.)
TE_0	$1.29297 - \text{i}\ 7.5205 \cdot 10^{-6}$	$1.29655 - \text{i}\ 6.4158 \cdot 10^{-6}$	1.2966

Bend mode phase shifts - I

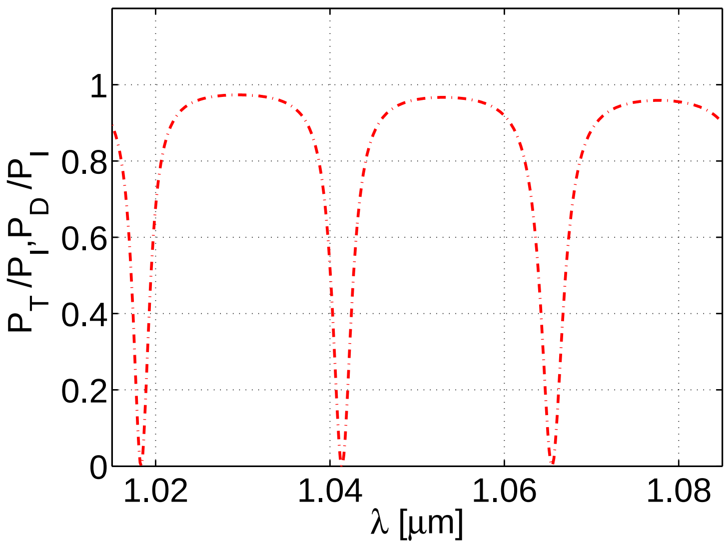


Dashed lines $= \beta/k_0$ *direct calculations*

Dots $=$ *Reference points* ($\epsilon_c = (1.5)^2$ *and* $\epsilon_c = (1.75)^2$)

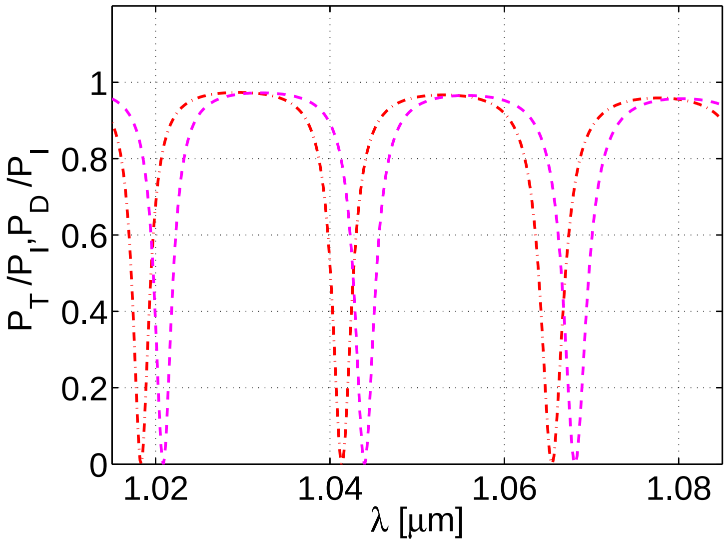
Solid segments$= \beta_{ref} + \delta\epsilon_c \frac{\partial\beta}{\partial\epsilon_c}\Big|_{ref}$, *perturbational evolution of* $\frac{\partial\beta}{\partial\epsilon_c}$

Microresonator spectrum perturbational evaluation - I



$\xrightarrow[\text{calculations}]{\text{Direct}}$ Unperturbed microresonator ($n_c = 1.5$)

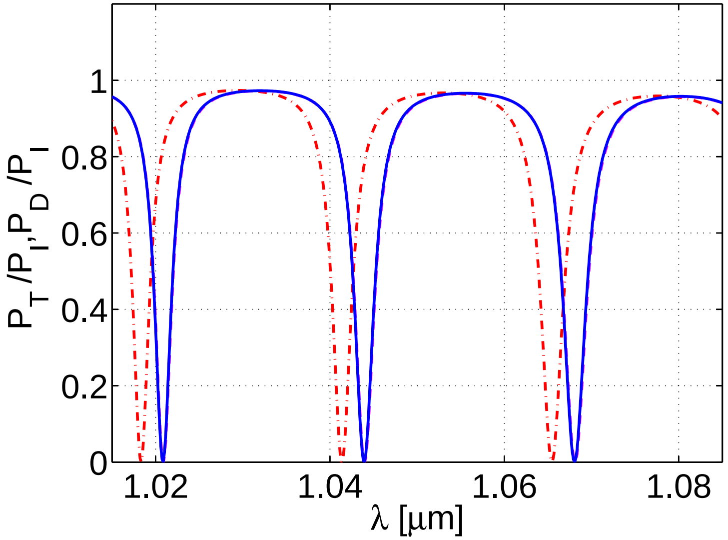
Microresonator spectrum perturbational evaluation - I



$\xrightarrow[\text{calculations}]{\text{Direct}}$ Unperturbed microresonator ($n_c = 1.5$)

$\xrightarrow[\text{calculations}]{\text{Direct}}$ Perturbed microresonator ($n_c = 1.504$)

Microresonator spectrum perturbational evaluation - I



Direct calculations → Unperturbed microresonator ($n_c = 1.5$)

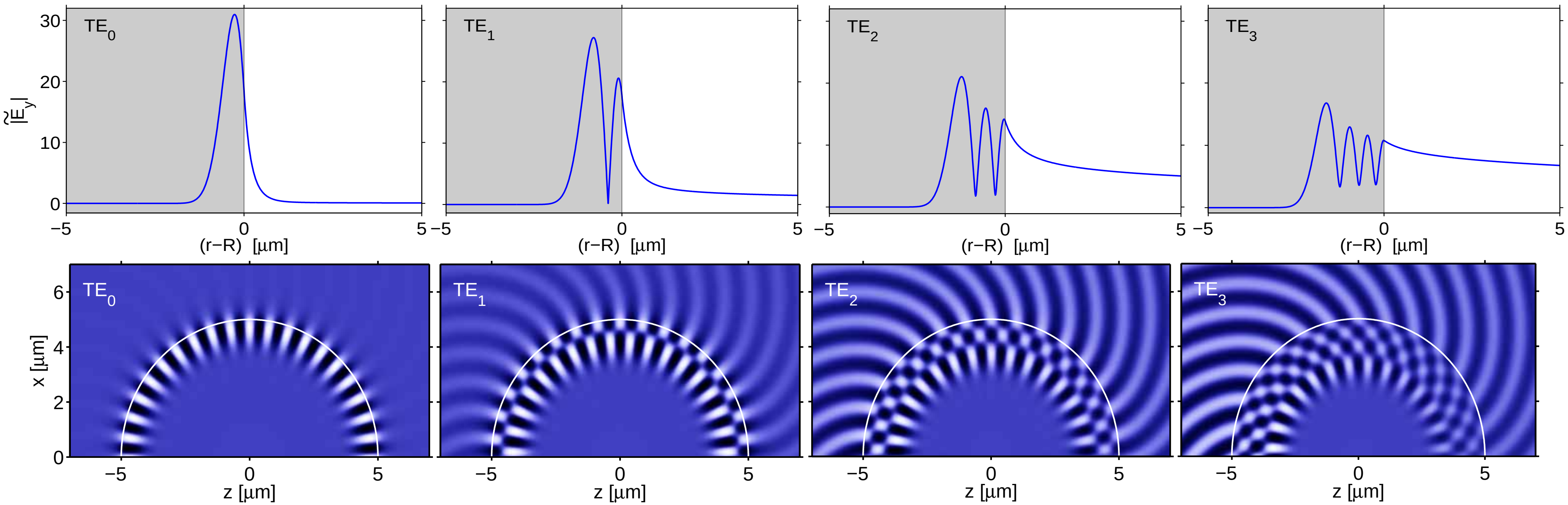
Direct calculations → Perturbed microresonator ($n_c = 1.504$)

perturbation expression based → Perturbed microresonator ($n_c = 1.504$)

- Resonance positions for perturbed microresonator:

	TE_0 resonances		
Direct simulation $n_c = 1.504$	1.021	1.044	1.068
Direct simulation $n_c = 1.5$ + Perturbational $\Delta\lambda$	1.021	1.044	1.068

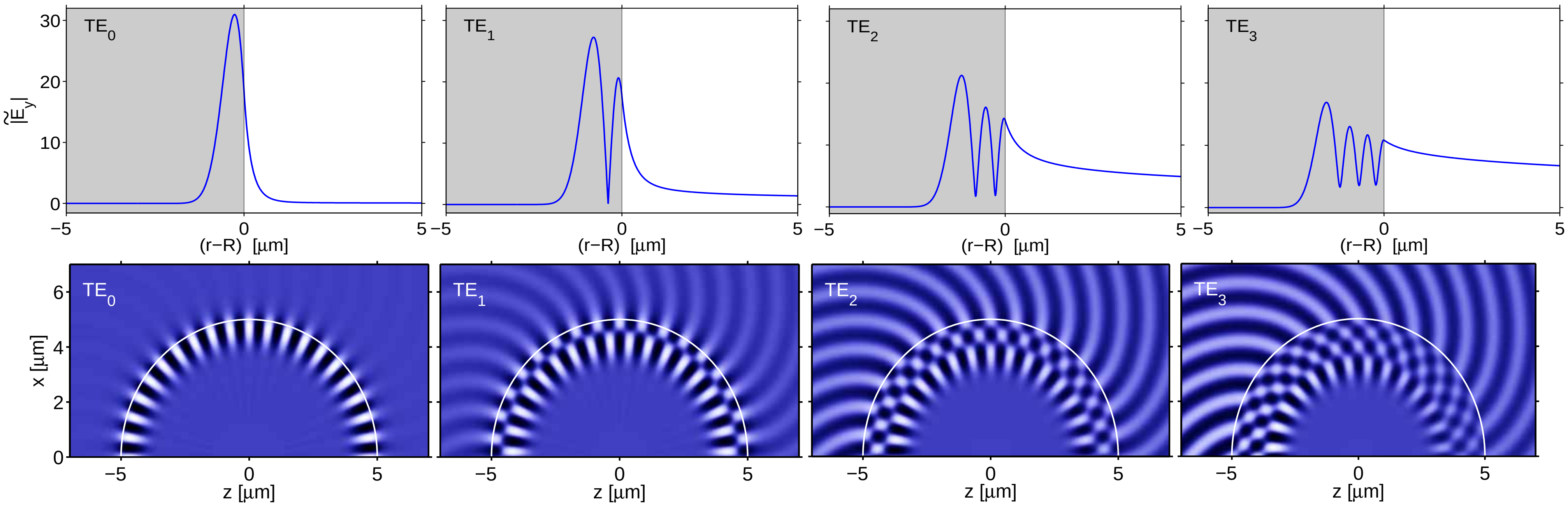
Whispering gallery modes



$|\tilde{E}_y|$ and real physical E_y field for a cavity with $n_c = 1.5$, $n_b = 1.0$, $R = 5\,\mu m$, $\lambda = 1.05\,\mu m$.

	$n_c = 1.5$
	N_{eff} (Direct)
TE_0	$1.32793 - \text{i}\ 9.53121 \cdot 10^{-7}$
TE_1	$1.16931 - \text{i}\ 4.03162 \cdot 10^{-4}$
TE_2	$1.04222 - \text{i}\ 5.74103 \cdot 10^{-3}$
TE_3	$0.92474 - \text{i}\ 1.31287 \cdot 10^{-2}$

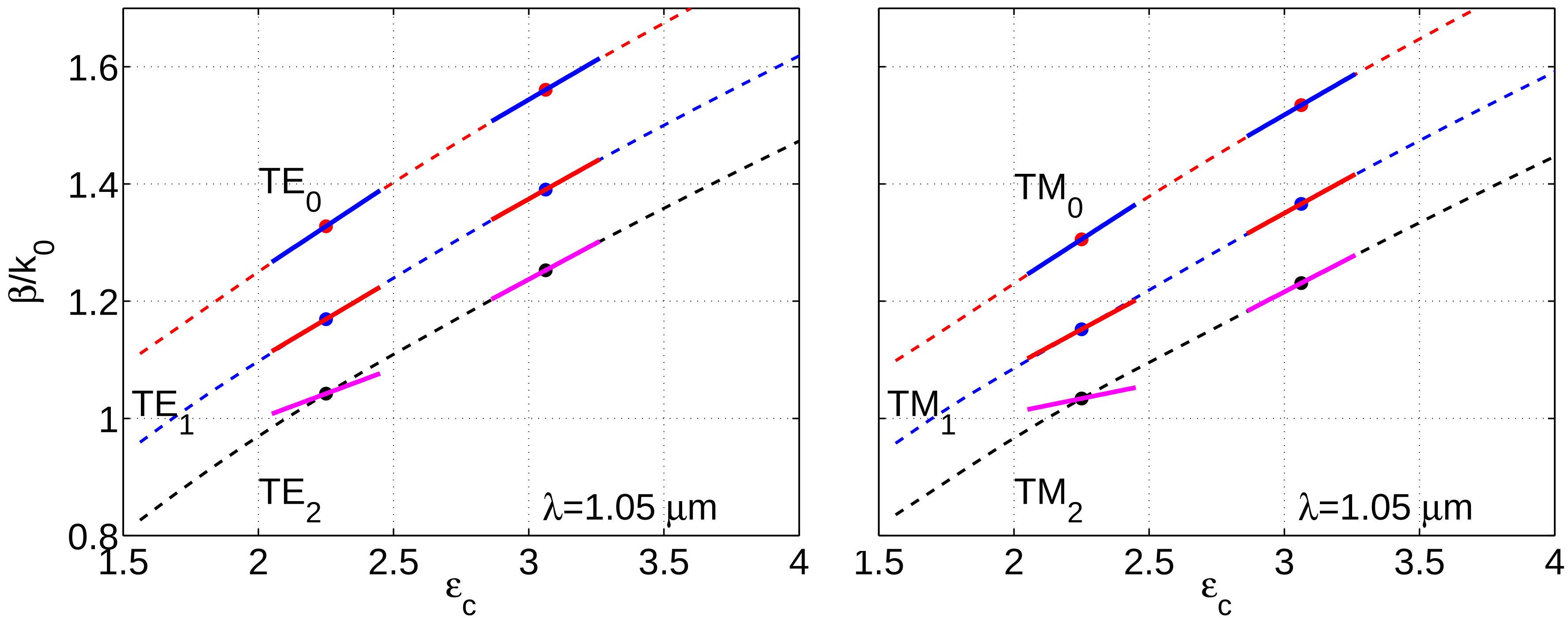
Whispering gallery modes



$|\tilde{E}_y|$ and real physical E_y field for a cavity with $n_c = 1.504$, $n_b = 1.0$, $R = 5\,\mu m$, $\lambda = 1.05\,\mu m$.

	$n_c = 1.5$	$n_c = 1.504$	
	N_{eff} (Direct)	N_{eff} (Direct)	$\Re(N_{\text{eff}})$ (Perturb.)
TE_0	$1.32793 - \text{i}\ 9.53121 \cdot 10^{-7}$	$1.33160 - \text{i}\ 8.01245 \cdot 10^{-7}$	1.33161
TE_1	$1.16931 - \text{i}\ 4.03162 \cdot 10^{-4}$	$1.17273 - \text{i}\ 3.61133 \cdot 10^{-4}$	1.17260
TE_2	$1.04222 - \text{i}\ 5.74103 \cdot 10^{-3}$	$1.04555 - \text{i}\ 5.46820 \cdot 10^{-3}$	1.04429
TE_3	$0.92474 - \text{i}\ 1.31287 \cdot 10^{-2}$	$0.92816 - \text{i}\ 1.28771 \cdot 10^{-2}$	0.92616

Bend mode phase shifts - II

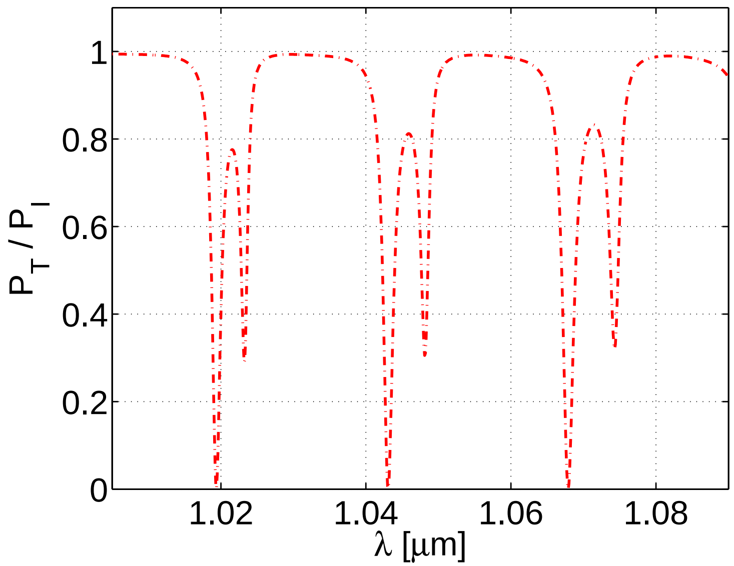


Dashed lines $= \beta/k_0$ *direct calculations*

Dots $=$ *Reference points* ($\epsilon_c = (1.5)^2$ *and* $\epsilon_c = (1.75)^2$)

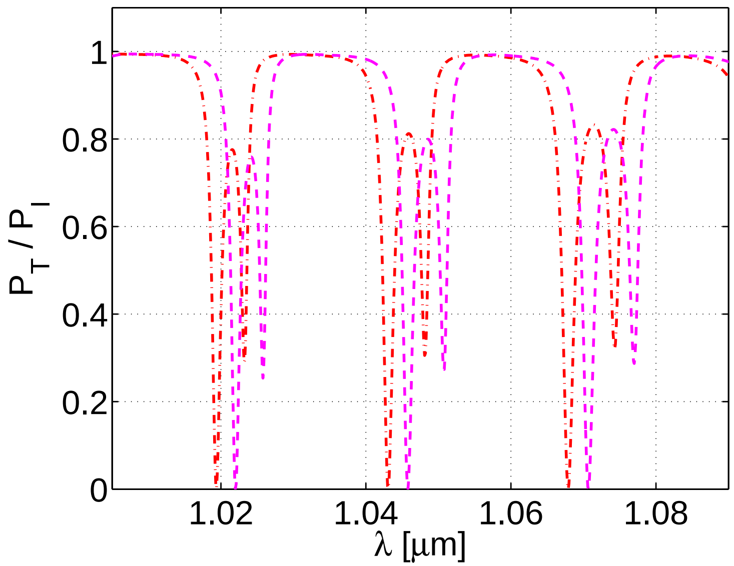
Solid segments$= \beta_{ref} + \delta\epsilon_c \frac{\partial\beta}{\partial\epsilon_c}\Big|_{ref}$, *perturbational evolution of* $\frac{\partial\beta}{\partial\epsilon_c}$

Microresonator spectrum perturbational evaluation - II



$\xrightarrow[\text{calculations}]{\text{Direct}}$ Unperturbed microresonator ($n_c = 1.5$)

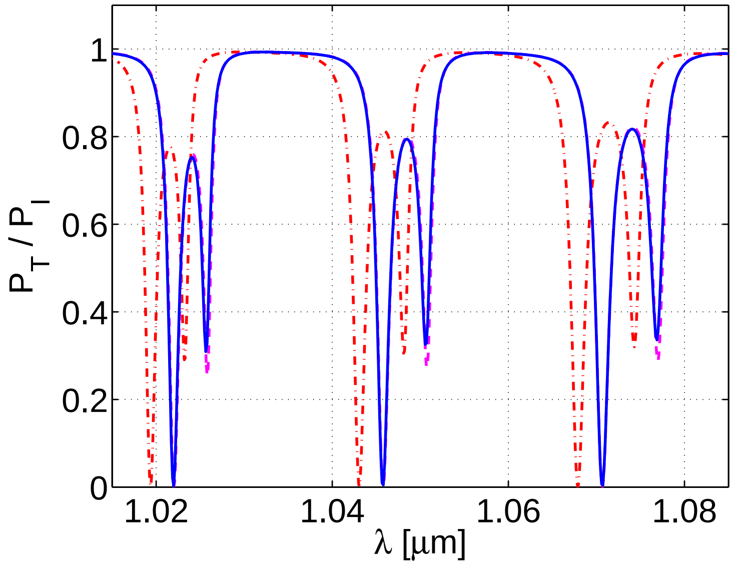
Microresonator spectrum perturbational evaluation - II

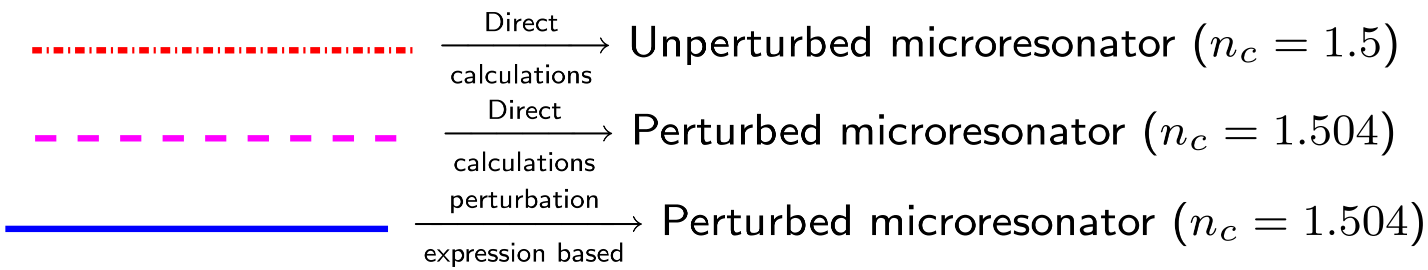


$\xrightarrow[\text{calculations}]{\text{Direct}}$ Unperturbed microresonator ($n_c = 1.5$)

$\xrightarrow[\text{calculations}]{\text{Direct}}$ Perturbed microresonator ($n_c = 1.504$)

Microresonator spectrum perturbational evaluation - II





- Resonance positions for perturbed microresonator:

	TE_0 resonances			TE_1 resonances		
Direct simulation $n_c = 1.504$	1.022	1.026	1.046	1.051	1.071	1.077
Direct simulation $n_c = 1.5$ + Perturbational $\Delta\lambda$	1.022	1.026	1.046	1.051	1.071	1.077

Conclusions

- $\surd$ An expression for shifts in the bend mode phase constants due to localised perturbations of the permittivity distribution is derived.
- $\surd$ Using this expression, shifts of resonance wavelengths of microresonators due to perturbation of the cavity core refractive index are calculated.
- $\surd$ Shifts of the spectral response as predicted by the perturbational expression agree very well with the directly computed spectral response.

:-)