Tight formation flying and sphere packing

Yoonsoo Kim, Da-Wei Gu and Ian Postlethwaite

Abstract

In this paper, we introduce a method for finding a tight formation of unmanned air vehicles (UAVs) via the classical sphere packing scheme. We first translate the tight formation finding problem to the problem of maximizing the second smallest eigenvalue $\lambda_2(G)$ of the graph Laplacian L_G. We then show how close the formation G_s obtained from the sphere packing scheme is to the optimal formation G^* that maximizes $\lambda_2(G)$. We show that $\lambda_2(G^*)/\lambda_2(G_s)$ is relatively small when the communication strength between two UAVs decays slowly with the distance between the two UAVs. This result implies that G_s can serve as a certificate that allows every graph to be quantitatively compared to G^*. In the light of this tight formation result, a modelling technique is given for the optimal airborne refuelling of multiple UAVs.

Key words: tight formation; sphere packing; algebraic connectivity; optimal refuelling; multiple UAVs

I. Introduction

In this paper, we are interested in the *tightest* formation of a group of multiple UAVs. Mathematically, we look for a formation G^* which maximizes the second smallest eigenvalue of the Laplacian matrix associated with G^*. It is well known that tight formation flying is practically advantageous in terms of reducing drag on UAVs, facilitating inter-vehicle communication, avoiding military radars, etc. In the last decade, many researchers have quantified tight formations through mathematical graph theory, e.g. [15]. The main idea is to consider each UAV x_i as a node

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v_i and to assign a number w_{ij} on the undirected edge e_{ij} between two nodes (v_i and v_j) based on their communication (connection) strength. This simple translation forms a graph $G(\mathcal{V}, \mathcal{E})$, where $v_i \in \mathcal{V}$ and $e_{ij} \in \mathcal{E}$. Interestingly, it is observed that the second smallest eigenvalue $\lambda_2(L_G)$ of the graph Laplacian L_G offers a good measure of tightness of a formation (graph) G. This observation is due to several facts that (i) $\lambda_2(G)$ is a classical connectivity measure of a graph G [3], [5], [14]; (ii) $\lambda_2(G)$ provides a sharp bound on both the *diameter* (denoted by $diam(G)$) and the minimum *degree* (denoted by $\delta(G) \stackrel{\text{def}}{=} \min_i \delta_i(G)$) of a graph G, where $diam(G)$ is the farthest distance between two UAVs in G and $\delta_i(G)$ is the sum of communication strengths between the ith UAV and all the other UAVs [14], [16]; (iii) $\lambda_2(G)$ is a measure of stability and robustness of networked dynamical systems [6], [17], [22].

In particular, when $\lambda_2(G)$ is considered in a state-dependent setting as proposed in [15], the characterization of distributed system states that maximize $\lambda_2(G)$ emerges as a natural problem. In this direction, as noticed in [11], there is only a handful of studies in the graph theory literature that are related to our eigenvalue maximization problem, e.g. [2]. Recently, numerical methods have been proposed to find a locally optimal formation via convex programming [7], [11]. However, these numerical approaches become computationally problematic as the number of nodes in G grows. For this reason, we are led in this paper to find a sub-optimal formation $\tilde{G}$ and a positive constant γ (≥ 1) such that

$$1 \leq \frac{\lambda_2(G^*)}{\lambda_2(\tilde{G})} \leq \gamma,$$

where G^* is the optimal formation. In other words, we are interested in finding a sub-optimal formation which is provably close to the optimal formation. We claim that a promising $\tilde{G}$ can be found by solving the well-known sphere packing problem: minimize the radius of a sphere that contains n spheres each with radius ρ. In the context of our study, the centre of each sphere could represent the position of a UAV and 2ρ is the minimal required distance between two UAVs to avoid collision. The paper is structured as follows. We provide a precise problem statement in §II and then propose an upper bound on γ in §III. This result is applied to the optimal airborne refuelling problem in §IV for multiple UAVs and a single tanker. Conclusions are given in §V.

II. Problem Statement

As stated earlier, we are interested in finding a graph (tight formation) which maximizes the second smallest eigenvalue of the associated graph Laplacian. A graph $G(x)$ (G, for short) represents a UAV formation $G(\mathcal{V}, \mathcal{E})$ in which each UAV's position x_i corresponds to a node v_i $(\in \mathcal{V})$ and the strength of communication between two UAVs (v_i and v_j) corresponds to the undirected edge weight w_{ij} $(\in \mathcal{E})$. The matrix L_G is a weighted graph Laplacian defined element-wise as

$$[L_G]_{ij} := \begin{cases} -w_{ij} & \text{if } i \neq j, \\ \sum_{s \neq i} w_{is} & \text{if } i = j, \end{cases} \tag{2.1}$$

and $\lambda_2(G)$ denotes the second smallest eigenvalue of the state-dependent Laplacian matrix L_G with its spectrum ordered as

$$\lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G).$$

Given these definitions, we consider the configuration problem

$$\mathbf{\Lambda}: \quad \max_{G \in \mathcal{G}_n^k} \quad \lambda_2(G), \tag{2.2}$$

where $x := [x_1^T, x_2^T, \ldots, x_n^T]^T \in \mathbf{R}^{2n}$ or $\mathbf{R}^{3n}$ is the vector of positions for n UAVs and $\mathcal{G}_n^k$ is the set of all feasible UAV configurations G in $\mathbf{R}^k$ $(k \in \{2, 3\})$ with n nodes. We note that any configuration $G \in \mathcal{G}_n^k$ satisfies the proximity constraint

$$d_{ij} := \|x_i - x_j\|^2 \geq 2\rho, \quad \text{for all } i \neq j, \tag{2.3}$$

preventing the elements from getting arbitrarily close to each other in their desire to maximize $\lambda_2(G)$ in (2.2).

From several well-known properties of the Laplacian matrix [8], $\lambda_2(G)$ can be written as

$$\begin{aligned} \lambda_2(G) &= \min_y y^T L_G y = \min_y \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} w_{ij}(y^T E_{ij} y) \\ &\stackrel{\text{def}}{=} \min_y \sum_{ij} w_{ij}(y^T E_{ij} y), \end{aligned} \tag{2.4}$$

where $y \in \mathcal{Y} \stackrel{\text{def}}{=} \{y \,|\, y \in \mathbf{R}^n, \; \|y\| = 1, \; y^T \mathbf{1} = 0\}$, $\mathbf{1} = [1, 1, \ldots, 1]^T \in \mathbf{R}^n$ and E_{ij} denotes the matrix with $(E_{ij})_{ii} = (E_{ij})_{jj} = 1$ and $(E_{ij})_{ij} = (E_{ij})_{ji} = -1$ and 0 at all other positions. Thus, $\mathbf{\Lambda}$ in (2.2) reads

$$\max_{G \in \mathcal{G}_n^k} \lambda_2(G) = \max_{w_{ij}} \min_y \sum_{ij} w_{ij}(y^T E_{ij} y) \stackrel{\text{def}}{=} \lambda_2(G^*). \tag{2.5}$$

For practical situations [19], we assume that the edge weight w_{ij} is a function of the distance d_{ij} between the ith and jth UAVs, i.e.

$$w_{ij} = \alpha d_{ij}^{-\beta},$$

where α, β are positive constants.[1]

As already stated, (2.5) is computationally a very difficult problem, especially for large n, so we aim to find a sub-optimal formation $\tilde{G}$ and a positive constant γ (≥ 1) such that

$$1 \leq \frac{\lambda_2(G^*)}{\lambda_2(\tilde{G})} \leq \gamma. \tag{2.6}$$

III. $\lambda_2(G^*)$ AND SPHERE PACKING

In this section, we offer a connection between $\lambda_2(G^*)$ and the classical sphere packing problem. The optimal sphere packing problem can be stated as follows: find the radius R^* of the smallest sphere that contains n identical spheres of radius ρ. We note that the optimal sphere packing problem reduces to the optimal circle packing problem when we consider finding a formation in a two-dimensional space. The optimal sphere packing problem itself is a famous open problem, but is known to be manageable for relatively small n. In particular, solutions are known to the optimal circle packing problem for $n \leq 20$ (see [9], [13]) and these are sufficient for our UAV application purposes. Fig. 1 shows a solution to the optimal circle packing problem when $n = 13$.

As each pair of UAVs must be separated by at least 2ρ, we consider each UAV as a sphere of radius ρ and obtain a seemingly tight formation by solving the optimal sphere packing problem. Our main goal here is to show how close the corresponding graph G_s to the optimal sphere packing is to the optimal (tightest) formation G^* that maximizes $\lambda_2(G)$, where $G \in \mathcal{G}_n^k$ ($k \in \{2, 3\}$ and $n \geq 2$). For this purpose, we start with a lower bound on $\lambda_2(G_s)$.

Theorem 3.1:

$$\lambda_2(G_s) \geq \alpha\, n\, (2^{-\beta})(R^* - \rho)^{-\beta}.$$

Proof: Recall from (2.5) that

$$\lambda_2(G_s) = \min_y \sum_{ij} w_{ij}(y^T E_{ij} y).$$

[1]We note that G is always *complete* in that $w_{ij} \neq 0$ for a finite d_{ij}. However, the problem under consideration is still nontrivial. See [11].

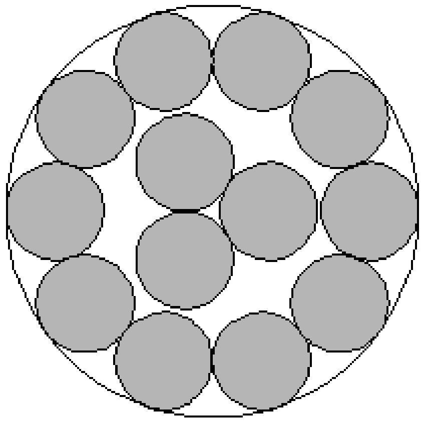

Fig. 1. A solution to the optimal circle packing problem when $n = 13$

Since $w_{ij} = \alpha d_{ij}^{-\beta}$ and $d_{ij} \leq 2(R^* - \rho)$,

$$\lambda_2(G_s) \geq \alpha(2(R^* - \rho))^{-\beta} \sum_{ij} y^T E_{ij} y.$$

The fact that $\sum_{ij} y^T E_{ij} y = n$ for any $y \in \mathcal{Y}$ proves the claim. ∎

This lower bound result is useful when the solution to the optimal sphere packing problem provides only the value of R^*, not actual G_s. Once G_s is known, one can directly compute $\lambda_2(G_s)$, instead of using the lower bound result.

Lemma 3.2: If $G_1, G_2 \in \mathcal{G}_n^k$ satisfy $w_{ij}^{G_1} \leq w_{ij}^{G_2}$ for all ij, where w_{ij}^G is the edge weight on edge e_{ij} of graph G. Then, $\lambda_2(G_1) \leq \lambda_2(G_2)$.

Proof: In (2.4), suppose y_1 and y_2 are the vectors that yield $\lambda_2(G_1)$ and $\lambda_2(G_2)$, respectively. Then

$$\begin{aligned} \lambda_2(G_2) &= \sum_{ij} w_{ij}^{G_2}(y_2^T E_{ij} y_2) \geq \sum_{ij} w_{ij}^{G_1}(y_2^T E_{ij} y_2) \\ &\geq \sum_{ij} w_{ij}^{G_1}(y_1^T E_{ij} y_1) = \lambda_2(G_1). \end{aligned}$$

∎

The proof of the following result adopts a similar proof to that used in [3].

Lemma 3.3: For a graph $G \in \mathcal{G}_n^k$, let G_θ be the graph formed by removing θ vertices and all adjacent edges to the removed vertices from G. Then

$$\lambda_2(G_\theta) \geq \lambda_2(G) - \theta\alpha(2\rho)^{-\beta}.$$

Proof: First obtain G_1 by removing a vertex v_1 and all adjacent edges to v_1 from G. Then define a new graph $\tilde{G}_1$ by reinserting the deleted edges from v_1 with edge weight $\alpha(2\rho)^{-\beta}$. We note that for a graph in $\mathcal{G}_n^k$, $\alpha(2\rho)^{-\beta}$ is the maximum edge weight because of (2.3). Then

$$L_{\tilde{G}_1} = \begin{bmatrix} L_{G_1} + \alpha(2\rho)^{-\beta} I & -\alpha(2\rho)^{-\beta}\mathbf{1} \\ -\alpha(2\rho)^{-\beta}\mathbf{1}^T & (n-1)\alpha(2\rho)^{-\beta} \end{bmatrix},$$

where I is the identity matrix with appropriate dimension. If y_1 is the eigenvector corresponding to $\lambda_2(G_1)$, then

$$L_{\tilde{G}_1} \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = [\lambda_2(G_1) + \alpha(2\rho)^{-\beta}] \begin{bmatrix} y_1 \\ 0 \end{bmatrix},$$

which implies that $\lambda_2(G_1) + \alpha(2\rho)^{-\beta}$ is a nonzero eigenvalue of $L_{\tilde{G}_1}$ and thus

$$\lambda_2(\tilde{G}_1) \leq \lambda_2(G_1) + \alpha(2\rho)^{-\beta}.$$

By Lemma 3.2, we have

$$\lambda_2(G) \leq \lambda_2(\tilde{G}_1) \leq \lambda_2(G_1) + \alpha(2\rho)^{-\beta}.$$

The claim follows by iteratively applying the above argument θ times. ■

Lemma 3.4: For a graph $G \in \mathcal{G}_n^k$, suppose that there exists a vertex v_i such that the edge weight $w_{ij} \leq \theta$ for all j $(j = 1, 2, \ldots, n,\ j \neq i)$. Then,

$$\lambda_2(G) \leq \theta n.$$

Proof: Recalling the definition (2.4) of $\lambda_2(G)$, let $y = -e_i/\sqrt{n(n-1)}$ in the righthand side of (2.5), where e_i is a vector of ones except for the ith entry which is equal to $1-n$. Note that the vector $-e_i/\sqrt{n(n-1)}$ belongs to the set $\mathcal{Y}$. Then,

$$\lambda_2(G) \leq \sum_{jk} w_{jk}(e_i^T E_{jk} e_i) \leq \theta \sum_{k(\neq i)} e_i^T E_{ik} e_i = \theta n.$$

■

This leads to the following upper bound on $\lambda_2(G)$ for any $G \in \mathcal{G}_n^k$.

Theorem 3.5: For a graph $G \in \mathcal{G}_n^k$

$$\lambda_2(G) \leq \alpha(n - \lceil n/2 \rceil)((\sqrt{2}R^*/2) - \rho)^{-\beta} + \alpha\lceil n/2 \rceil(2\rho)^{-\beta}.$$

Proof: For a graph $G(\mathcal{V}, \mathcal{E}) \in \mathcal{G}_n^k$, we consider a sphere $\mathbf{s}_i$ $(i = 1, 2, \ldots, n)$ with centre $v_i \in \mathcal{V}$ and radius ρ. We then consider the smallest sphere $\mathbf{S}$ that contains all such spheres $\mathbf{s}_i$

and suppose the radius of $\mathbf{S}$ is R. Clearly there exists at least one $\mathbf{s}_i$ touching $\mathbf{S}$, and let us call the touching point p. We then consider another sphere $\mathbf{S}_1$ (respectively, $\mathbf{S}_2$) with centre p and radius $\tilde{R}/2$ (respectively, $\tilde{R}$), where $\tilde{R} = \sqrt{2}R$, as seen in Fig. 2. We note that the intersected region of $\mathbf{S}$ and $\mathbf{S}_2$, i.e. $\mathbf{S} \cap \mathbf{S}_2$, has a diameter[2] of $2R$. Thus, the construction of $\mathbf{S}$ implies that there exists at least one $\mathbf{s}_j$ such that the set $\mathbf{s}_j \cap (\mathbf{S} \backslash int(\mathbf{S} \cap \mathbf{S}_2))$ is not empty, where $\mathbf{A} \backslash \mathbf{B}$ denotes the difference between two sets $\mathbf{A}$ and $\mathbf{B}$, and $int(\mathbf{A})$ the interior of a set $\mathbf{A}$. In fact, if the set $\mathbf{s}_j \cap (\mathbf{S} \backslash int(\mathbf{S} \cap \mathbf{S}_2))$ was empty, one could construct $\mathbf{S}_2$ such that $\mathbf{S} \cap \mathbf{S}_2$ has a smaller diameter ($< 2R$), and this contradicts the fact that $\mathbf{S}$ is the smallest sphere containing $\mathbf{s}_1, \ldots, \mathbf{s}_n$.

We now prove the claim for two cases: at least $\lceil n/2 \rceil$ spheres $\mathbf{s}_j$ such that $v_j \in \mathbf{S} \backslash int(\mathbf{S} \cap \mathbf{S}_1)$ or $v_j \in int(\mathbf{S} \cap \mathbf{S}_1)$, where $\lceil x \rceil$ is the least integer greater than x. For the first case, we consider the graph $\tilde{G}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}})$ that is obtained by removing $\lceil n/2 \rceil$ vertices v_l from G, where the v_l include all the vertices contained in $int(\mathbf{S} \cap \mathbf{S_1})$ except v_i. Then, by Lemma 3.3,

$$\lambda_2(G) \leq \lambda_2(\tilde{G}) + \lceil n/2 \rceil \alpha (2\rho)^{-\beta}.$$

On the other hand, note that $\|v_i - v_{l'}\| \geq (\tilde{R}/2) - \rho = (\sqrt{2}R/2) - \rho$, where $l' \in \widetilde{\mathcal{V}}$ $(l' \neq i)$. In view of Lemma 3.4, we have

$$\lambda_2(\tilde{G}) \leq (n - \lceil n/2 \rceil) \alpha ((\sqrt{2}R/2) - \rho)^{-\beta}.$$

The claim then follows from the fact that $R^* \leq R$.

For the second case, one can obtain the same result by letting v_j play the similar role as v_i in the first case. This time we consider the graph $\tilde{G}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}})$ that is obtained by removing $\lceil n/2 \rceil$ vertices v_l from G, where the v_l include all the vertices contained in $\mathbf{S} \backslash int(\mathbf{S} \cap \mathbf{S_1})$ except v_j, and note that $\|v_j - v_{l'}\| \geq (\tilde{R}/2) - \rho = (\sqrt{2}R/2) - \rho$, where $l' \in \widetilde{\mathcal{V}}$ $(l' \neq j)$. ■

Our main result is a simple combination of Theorems 3.1 and 3.5 with $R^* = \kappa\rho$ ($\kappa \geq 2$).

Corollary 3.6:

$$\begin{aligned} 1 &\leq \frac{\lambda_2(G^*)}{\lambda_2(G_s)} \\ &\leq \frac{\lceil n/2 \rceil}{n} (\kappa - 1)^\beta + \left(1 - \frac{\lceil n/2 \rceil}{n}\right) \left(\frac{2(\kappa - 1)}{\sqrt{2}\kappa/2 - 1}\right)^\beta \\ &\stackrel{\text{def}}{=} \gamma. \end{aligned} \tag{3.7}$$

[2]The diameter of a set (region) $\mathbf{A}$ is the maximum distance between two elements (points) in $\mathbf{A}$.

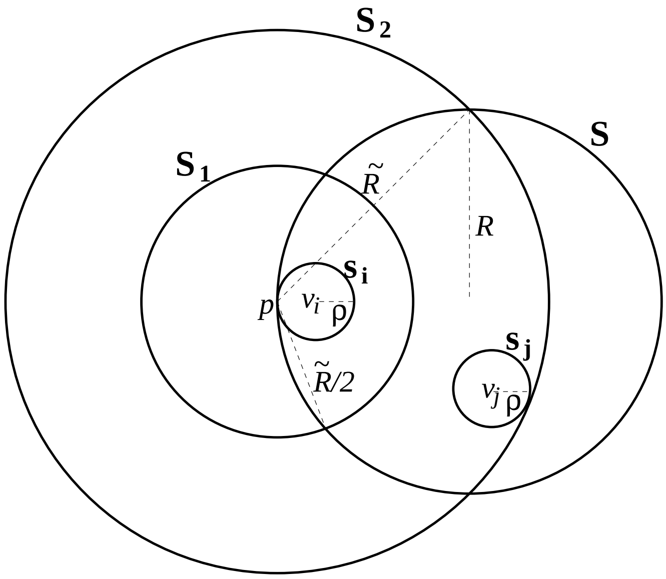

Fig. 2. Construction of spheres $\mathbf{S}$, $\mathbf{S}_1$ and $\mathbf{S}_2$

The lower bound γ on the ratio of $\lambda_2(G^*)$ to $\lambda_2(G_s)$ is a function of only two parameters, β and κ, and becomes large as the two parameters do likewise. This implies that the upper bound (3.7) is valid only if the communication strength between two UAVs decays slowly with the distance between the two UAVs, i.e. β is small. From the known values of R^* in [1] for $k = 2$ and $3 \leq n \leq 20$, we tabulate the values of γ for $\rho = 1$ and $\beta \in \{1, 2, 3\}$ in Table 1. However, as addressed before, we could directly compute $\lambda_2(G_s)$ and improve γ, instead of using the lower bound result given in Theorem 3.1. In other words, if $\lambda_2(G_s) = \alpha\eta_s$, we have

$$\begin{aligned} \frac{\lambda_2(G^*)}{\lambda_2(G_s)} &\leq \frac{(n - \lceil n/2 \rceil)((\sqrt{2}\kappa/2) - 1)^{-\beta} + \lceil n/2 \rceil 2^{-\beta}}{\eta_s \rho^\beta} \\ &\stackrel{\text{def}}{=} \gamma_s. \end{aligned} \tag{3.8}$$

We first note that $\gamma_s = 1$ when $\beta = 0$ and thus $\eta_s = n$, i.e. every pair of UAVs have perfect communication regardless of the distance between them. For nonzero β, we tabulate the values of γ_s in Table II for $\rho = 1$ and various k, n and β. The values in Table II are based on the same G_s given in [1] whose η_s are tabulated in Table III. Table II clearly shows that γ_s is certainly better than γ and valid for larger β, although it is still observed that γ_s increases as β does likewise. Note that smaller β is likely to induce smaller γ_s because it makes a graph more complete. Fig. 3, a pictorial version of Table III for $\beta = 1$ and 2, suggests that for $\beta = 1$

TABLE I

THE VALUES OF κ AND γ FOR $k = 2$, $3 \le n \le 20$, $\rho = 1$ AND $\beta \in \{1, 2, 3\}$

n	3	4	5	6	7	8
κ	2.154	2.414	2.701	3.000	3.000	3.304
γ ($\beta = 1$)	2.240	2.707	2.516	2.784	2.671	2.876
γ ($\beta = 2$)	7.377	9.707	7.328	8.363	7.739	8.600
γ ($\beta = 3$)	29.65	33.42	23.86	26.70	24.03	26.62
n	9	10	11	12	13	14
κ	3.613	3.813	3.923	4.029	4.236	4.328
γ ($\beta = 1$)	2.946	3.065	3.092	3.153	3.234	3.279
γ ($\beta = 2$)	8.815	9.457	9.597	9.955	10.50	10.76
γ ($\beta = 3$)	26.79	29.37	29.89	31.48	34.00	35.29
n	15	16	17	18	19	20
κ	4.521	4.615	4.792	4.863	4.863	5.122
γ ($\beta = 1$)	3.374	3.405	3.502	3.516	3.534	3.633
γ ($\beta = 2$)	11.41	11.64	12.36	12.48	12.61	13.44
γ ($\beta = 3$)	38.65	39.92	43.93	44.72	45.40	50.56

γ_s is below 2.5 for n up to 20 and, in particular, for $\beta = 2$ γ_s assumes the minimum value of 3.8697 at $n = 7$. In order for a practical interpretation of this result, we consider a circle packing solution for $n = 7$, as shown in Fig. 4. If the parameters ρ, α and β are chosen as 10 (m), 20 and 2, respectively, one can easily expect that some pairs of *non-contacting* UAVs may lose their communication links because the corresponding communication strengths can be almost 100 times weaker than the maximum possible strength (=1). However, the sphere packing approach guarantees that the formation induced from the circle packing solution is at most less than four times as *lose* as the true optimal (two-dimensional) formation.

Another useful interpretation of (3.8) is that $G_s \in \mathcal{G}_n^k$ can serve as a certificate that allows every graph $G \in \mathcal{G}_n^k$ to be quantitatively compared to $G^* \in \mathcal{G}_n^k$. Namely, if $\lambda_2(G) = \alpha\eta$, we have

$$\frac{\lambda_2(G^*)}{\lambda_2(G)} \le \frac{(n - \lceil n/2 \rceil)((\sqrt{2}\kappa/2) - 1)^{-\beta} + \lceil n/2 \rceil 2^{-\beta}}{\eta\rho^\beta} \stackrel{\text{def}}{=} \tilde{\gamma}.$$

Since the upper bound result Theorem 3.5 is for a general graph $G \in \mathcal{G}_n^k$, $\tilde{\gamma}$ could be further improved by using an optimality condition on $\lambda_2(G^*)$.

TABLE II

THE VALUES OF γ_s FOR $\rho = 1, 3 \leq n \leq 20$ AND $\beta \in \{1, 2, 3\}$

n	3	4	5	6	7	8
$\beta = 1$	1.9399	2.2426	2.0439	2.2376	1.9761	2.1381
$\beta = 2$	5.5300	6.0000	4.4598	5.0175	3.8697	4.3004
$\beta = 3$	19.243	17.456	10.834	10.957	7.4354	8.0550
n	9	10	11	12	13	14
$\beta = 1$	2.1992	2.2373	2.2024	2.2024	2.2559	2.2708
$\beta = 2$	4.4074	4.4991	4.2889	4.2727	4.4394	4.4948
$\beta = 3$	7.9213	8.0064	7.2205	7.1455	7.4328	7.5649
n	15	16	17	18	19	20
$\beta = 1$	2.2886	2.3356	2.3154	2.3970	2.3488	2.4231
$\beta = 2$	4.5535	4.7444	4.6316	5.0089	4.7679	5.1148
$\beta = 3$	7.6761	8.1247	7.7384	8.7919	8.0595	9.0182

TABLE III

THE VALUES OF η_s FOR $k = 2, 3 \leq n \leq 20$ AND $\beta \in \{1, 2, 3\}$

n	3	4	5	6	7	8
$\beta = 1$	1.500	1.7071	1.8090	1.8660	2.3660	2.3349
$\beta = 2$	0.7500	0.7500	0.6910	0.6250	0.8750	0.7530
$\beta = 3$	0.3750	0.3384	0.2795	0.2284	0.3534	0.2699
n	9	10	11	12	13	14
$\beta = 1$	2.3066	2.4349	2.6415	2.8327	2.8845	3.0373
$\beta = 2$	0.6590	0.6641	0.7200	0.7617	0.7337	0.7561
$\beta = 3$	0.2132	0.2060	0.2278	0.2377	0.2193	0.2214
n	15	16	17	18	19	20
$\beta = 1$	3.1400	3.2258	3.3901	3.4167	3.6997	3.6374
$\beta = 2$	0.7577	0.7506	0.7886	0.7512	0.8416	0.7732
$\beta = 3$	0.2163	0.2080	0.2212	0.1985	0.2320	0.2001

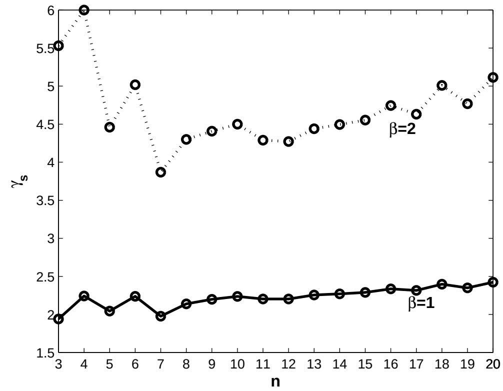

Fig. 3. γ_s versus n

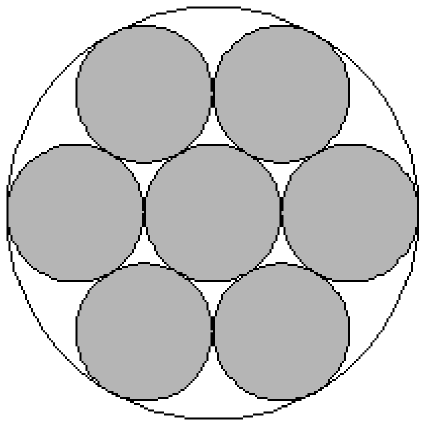

Fig. 4. A solution to the optimal circle packing problem when $n = 7$

IV. An Application: Optimal Schedules for the Airborne Refuelling of Multiple UAVs from a Single Tanker

In this section, we consider the optimal airborne refuelling problem for multiple UAVs and a single tanker.[3] To be precise, we are interested in the optimal refuelling schedule for n UAVs and a single tanker such that (i) all involved UAVs maintain minimal separation between neighbouring UAVs; (ii) the finally served UAV's waiting time is minimised; and (iii) the UAVs fly in tight

[3]There should be many other examples for which the proposed sphere packing heuristic can be applied. As one of anonymous reviewers suggests, one can imagine a group of mobile agents which desire to rendezvous in order to exchange information before re-deploying. Since each agent may occupy physical space, one may consider the proposed sphere-packing heuristic for the rendezvous formation for efficient information exchange.

formation which facilitates communication between UAVs throughout the refuelling process. The refuelling scheduling problem can be viewed as a variant of the multiple aircraft landing problem, although a tanker associated with the former is dynamic as opposed to a static landing strip (or an airport) associated with the latter. As a result, one can approach the refuelling problem using the way the multiple aircraft landing problem has been handled. Issues (i) and (ii) are extensively studied in the literature. For example, one can refer to an excellent survey [12] for aircraft conflict resolution and a recent paper [20] on aircraft landing scheduling. More closely related to our application, we note two works on satellite constellations [21] and multiple UAV refuelling [24].

We note, however, that most of the scheduling-related work on the refuelling problem and similar contexts neglect or treat the collision avoidance issue (i) in a trivial manner. It seems to be implicitly assumed that once a UAV (or aircraft) knows it is to be served (or should land), each waiting UAV (or aircraft) flies or rotates along a fixed one-dimensional path, e.g. a circular orbit, whilst maintaining a minimum separation with the other UAVs. For instance, for a single tanker and n UAVs $x_i(t)$ (at positions x_i and time t), one can consider $n-1$ circular orbits concentric to the tanker. Each orbit has a distinct radius of a constant multiple of 2ρ, where $\|x_i(t) - x_j(t)\| \geq 2\rho$ for every t and $i,\, j \in \{1, 2, \ldots, n\}$ $(i \neq j)$, and contains exactly one UAV constantly rotating along it. This so-called multiple-orbit approach is clearly optimal in terms of the issues (i) and (ii), as long as all waiting UAVs simultaneously transfer down to their next orbit whenever the tanker starts refuelling a new UAV. However, we note that this multiple-orbit approach is feasible only if all waiting UAVs maintain perfect communication between each other. Furthermore, it is desirable to reduce the number of orbits in order for UAVs not to spend too much fuel rotating around large orbits.

The communication issue (iii) becomes more necessary in this multiple UAV refuelling context. Typical UAV missions in which no human operators are involved are performed in an uncertain environment, and therefore it is absolutely essential that communication between UAVs is as secure as possible. As an example, UAVs must have the capability of deciding which UAV should be served first based on the amount of fuel remaining. For faster decision making, it is desired that the distance between each pair of UAVs is minimized. On the other hand, each UAV is required to maintain clear connections with as many other UAVs as possible, in case it loses one (or more) of the connections for some reason. For these purposes, we incorporate the tight formation flying concept into the previous orbit model to resolve all the aforementioned three issues.

Our improved orbit approach is briefly sketched as follows. We first define a two-dimensional plane $\mathbf{P}$ containing the tanker and all the UAVs. The plane $\mathbf{P}$ contains a single orbit $\mathbf{O}$ surrounding the tanker, and all waiting UAVs form the two-dimensional formation G_s $(\in \mathcal{G}^2_{n-1})$, obtained by solving the optimal sphere packing problem, and rotate together along $\mathbf{O}$. A UAV leaving the formation G_s to be refuelled uses the space above $\mathbf{P}$. After refuelling the UAV leaves the tanker using the space below $\mathbf{P}$. We assume that UAVs are capable of changing to another formation G_s $(\in \mathcal{G}^2_{n'})$ whenever the number of UAVs in formation changes to n'. In this way, one can resolve all the three issues (i), (ii) and (iii) efficiently.

V. Concluding Remarks

We have introduced a method for finding a tight formation motivated by sphere packing. We have interpreted the tightest formation G^* in the context of the graph Laplacian L_{G^*}, and have derived an analytical upper bound on the ratio γ of the second smallest eigenvalue of L_{G^*} to that of L_{G_s} obtained by solving the simpler sphere packing problem. We showed that γ is relatively small when the communication strength between two UAVs decays slowly with the distance between the two UAVs. This result implies that G_s can serve as a certificate that allows every graph to be quantitatively compared to G^*. The tight formation result can then be used to determine a strategy for the refuelling of multiple UAVs.

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