[7] A. G. Mazko, "The Lyapunov matrix equation for a certain class of regions bounded by algebraic curves," *Sov. Automat. Control*, vol. 13, pp. 37–42, 1980.
[8] J. B. Lasserre, "Polynomials with all zeros real and in a prescribed interval," *J. Alg. Comb.*, vol. 16, pp. 231–237, 2002.
[9] H. Taub and S. Gutman, "A symmetric matrix criterion for polynomial root clustering," *IEEE Trans. Circuits Syst.*, vol. 37, pp. 243–248, Feb. 1990.
[10] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM Rev.*, vol. 38, pp. 49–95, 1996.

Quadratically Constrained Attitude Control via Semidefinite Programming

Yoonsoo Kim and Mehran Mesbahi

***Abstract*—We consider the attitude control problem augmented with an arbitrary number of nonconvex quadratic constraints. By utilizing the implicit magnitude constraint of the state vector (i.e., quaternion), we are able to reduce this problem to a convex quadratically constrained quadratic program or a semidefinite program. This result leads to an efficient solution strategy for general quadratically constrained attitude control problems. An example demonstrates the proposed methodology.**

***Index Terms*—Attitude control, convex optimization, semidefinite programming, state constrained control.**

I. Introduction

Consider a space mission with the task of collecting images of distant stars or other heavenly bodies. In such a mission, the spacecraft is often required to change its orientation (attitude) in some optimal manner while ensuring that its sensitive optical instruments are not exposed to bright objects in the sky such as the sun. An important control problem in such a setting is thus to determine the necessary control torques such that the cones emanating from these sensitive optical devices exclude the bright object(s) during the reorientation maneuver. This problem, which we refer to as the quadratically constrained attitude control (Q-CAC), can in principle be formulated as an optimal nonlinear control problem subject to nonconvex quadratic constraints.

The Q-CAC problem has arisen in the context of many space science missions [1]. Nonetheless, an efficient solution strategy for the Q-CAC has not been available so far—in fact, this problem is often perceived to be computationally difficult and potentially in the class of NP-hard problems.[1] Meanwhile, the two sources of problem complexity—nonlinearity of the attitude dynamics and nonconvexity of the associated quadratic constraints—have often been considered separately. In this venue, the goal has been to solve either the nonconvex quadratically constrained control problem for linear plants [7], or the linearly constrained control problems for nonlinear plants [4], [6], [14], [16], or the more promising linearly constrained control problems for linear plants [5], [10]. We particularly note the work of Yakubovich [17] on the quadratically constrained optimal control problems and employing the S-procedure and the multiplier method as the core of its solution strategy. Motivated by the new advances in the computational methods for solving optimization problems defined over matrix spaces, a wide range of control problems have recently been approached via the theory of linear matrix inequalities (LMIs) and semidefinite programming (SDP) [3], [15]. Specifically, we note the successful LMI approaches in resolving optimization problems with limited number of quadratic constraints, as reported, for example, in [2] and [13].

In this note, we employ an SDP approach for the general Q-CAC problem, by including, and in fact exploiting, both the nonlinear and nonconvex features in the problem. Our main contribution is an SDP-based solution strategy (or one based on convex quadratically constrained quadratic program) for Q-CACs augmented with an *arbitrary* number of nonconvex quadratic constraints. In retrospect, we believe that the underlying ideas of our approach will contribute to the development of new solution methodologies for a much wider class of constrained control problems beyond that of the attitude control.

The outline of the paper is as follows. In Section II, we formalize the Q-CAC problem; Section III is devoted to the main result, where we present the general convexification procedure for handling nonconvex quadratic attitude constraints. We then proceed to compute the required control torques via an SDP-based procedure for the discretized Q-CAC problem. A numerical example in Section IV concludes our presentation.

First, a few words on the notation. The space of real numbers, real n dimensional vectors, $m \times n$ real matrices, $n \times n$ symmetric matrices, and $n \times n$ positive–semidefinite matrices, will be denoted by $\mathbf{R}, \mathbf{R}^n, \mathbf{R}^{m\times n}, \mathcal{S}^n$, and $\mathcal{S}^n_+$, respectively. I_n and $0_{m\times n}$ denote the $n \times n$ identity and the $m \times n$ zero matrices; $\mathbf{Diag}\,(x)$ will designate the square diagonal matrix of appropriate dimensions with the vector x on its diagonal. The notation $\|q\|$ designates the two-norm of vector $q \in \mathbf{R}^n$; $\langle A, B\rangle$ is the inner product for square symmetric matrices A and B, defined as the trace of their matrix product.

II. Problem Statement

The formal Q-CAC problem considered in this paper is as follows[2]: Find the control torques $u_i(t) \in \mathbf{R}^3 (i = 1, 2, 3)$, over the time interval $t \in [t_0, t_f]$, subject to the initial and the terminal conditions on the angular velocity vector $\omega(t)$ and the attitude quaternion $q(t)$

$$(\omega(t_0), q(t_0)) \quad (\omega(t_f), q(t_f))$$

with

$$\omega(t) = [\omega_1(t), \omega_2(t), \omega_3(t)]^T$$
$$q(t) = [q_1(t), q_2(t), q_3(t), q_4(t)]^T$$

the rigid body equations of motion

$$J_1\dot{\omega}_1(t) - (J_2 - J_3)\omega_2(t)\omega_3(t) = u_1(t) \tag{2.1}$$
$$J_2\dot{\omega}_2(t) - (J_3 - J_1)\omega_3(t)\omega_1(t) = u_2(t) \tag{2.2}$$
$$J_3\dot{\omega}_3(t) - (J_1 - J_2)\omega_1(t)\omega_2(t) = u_3(t) \tag{2.3}$$

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[1]The researchers and engineers who *need* to solve this class of problems have nevertheless proposed creative solution strategies for their solution [1].

[2]As stated, Q-CAC is a state constrained, rather than an optimal, control problem.

(J_i being the body inertia along the principle axis i), boundedness of the angular velocities and control torques

$$|\omega_i(t)| \le \gamma_1 \quad \text{and} \quad |u_i(t)| \le \gamma_2, \qquad i = 1,2,3, \quad t \in [t_0, t_f]$$

(given the constants $\gamma_1, \gamma_2 > 0$), norm preserving kinematic constraint

$$\dot{q}(t) = \frac{1}{2}\Omega(t)q(t) \tag{2.4}$$

with

$$\Omega(t) = \begin{bmatrix} 0 & \omega_3(t) & -\omega_2(t) & \omega_1(t) \\ -\omega_3(t) & 0 & \omega_1(t) & \omega_2(t) \\ \omega_2(t) & -\omega_1(t) & 0 & \omega_3(t) \\ -\omega_1(t) & -\omega_2(t) & -\omega_3(t) & 0 \end{bmatrix}$$

guaranteeing that

$$\|q(t)\| = 1, \quad \text{for} \quad t \in [t_0, t_f]$$

and, finally, the attitude constraints

$$q(t)^T \tilde{A}_i(x, y, \theta) q(t) \le 0, \quad \text{for} \quad i = 1, \ldots, m \tag{2.5}$$

with

$$\tilde{A}_i(x, y, \theta) = \tilde{A}_i = \begin{bmatrix} A_i & b_i \\ b_i^T & d_i \end{bmatrix} \in \mathbf{R}^{4\times 4} \tag{2.6}$$

and

$$A_i := x_i y_i^T + y_i x_i^T - \left(x_i^T y_i + \cos\theta\right) I_3$$
$$b_i := x_i \times y_i \quad d_i := x_i^T y_i - \cos\theta$$

see Fig. 1. To see how constraints of the form (2.5) specify the exclusion zones in the attitude space-cones emanating from the spacecraft's sensitive instruments that need to exclude the bright objects in the sky during the maneuver—one proceeds as follows [1]. First, consider the unit celestial vector x (specified in the inertial coordinates) and the unit body vector y (specified in the body coordinates).[3] We would like the time evolution of the cone with a half-angle θ around the inertially represented vector y, y_I, to exclude x at all times. Enforcing the inequality

$$x^T y_I \le \cos\theta, \qquad \theta \in (0, \pi) \tag{2.7}$$

guarantees that in fact a minimum angular separation θ is maintained between the two vectors x and y_I (see Fig. 2).[4] Inequality (2.7) is now translated in terms of the constraints on the orientation of the spacecraft. Since y_I represents y in the inertial coordinate frame in (2.7) it satisfies

$$y_I = y - 2\left(q_0^T q_0\right) y + 2\left(q_0^T y\right) q_0 + 2q_4(y \times q_0) \tag{2.8}$$

where the spacecraft attitude quaternion is represented by

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ -- \\ q_4 \end{bmatrix} = \begin{bmatrix} q_0 \\ -- \\ q_4 \end{bmatrix}$$

see, for example, [8] and [11]. We now expand the quadratic form (2.5)

$$2\left(q_0^T y\right)\left(q_0^T x\right) - \left(q_0^T q_0\right)(x^T y) + q_4^2 (x^T y) + 2q_4 q_0^T (x \times y) \le \left(q_0^T q_0 + q_4^2\right)\cos\theta$$

[3]For notational brevity, we will drop the subscript i.

[4]Note that considering the case where $\theta \in [\pi, 2\pi]$ is not necessary.

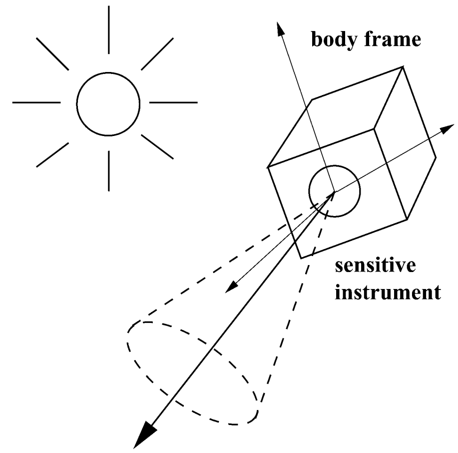

Fig. 1. Sun-avoidance constraint.

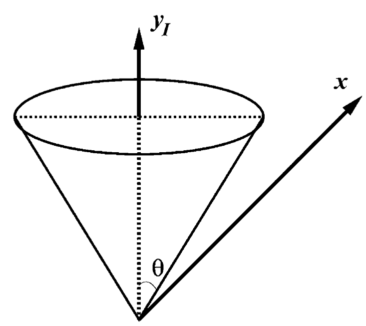

Fig. 2. Exclusion zone.

which can be written as

$$x^T y - 2\left(q_0^T q_0\right)(x^T y) + 2\left(q_0^T y\right)\left(q_0^T x\right) + 2q_4 (x \times y)^T q_0 \le \cos\theta.$$

In view of (2.8), this last expression is equivalent to (2.7).

III. Main Result

In this section, we will consider finding the required control torques in the Q-CAC problem of Section II. In this venue, we will first state and prove Proposition 3.1. We then proceed to discretize the kinematic and dynamic constraints (2.1)–(2.4). The implications of Proposition 3.1 in the context of the discrete Q-CAC problem is then explored.

Let

$$\mathcal{B}_\eta := \{x \in \mathbf{R}^n \mid \|x\| = \eta\} \tag{3.9}$$

where η is a nonnegative real number.

Proposition 3.1: Given the matrices $W_i \in \mathcal{S}^n, b_i \in \mathbf{R}, i = 1, 2, \ldots, m$, and $\eta > 0$, let

$$\mathcal{W} := \{x \in \mathcal{B}_\eta \mid x^T W_i x \ge b_i, \qquad i = 1, \ldots, m\} \tag{3.10}$$

be nonempty. Then a feasible element of $\mathcal{W}$ can be found by solving a magnitude constrained SDP.

Proof: Our strategy is to convexify the quadratic constraints

$$x^T W_i x \ge b_i, \quad \text{for} \quad i = 1, \ldots, m.$$

For $i \in \{1, \ldots, m\}$, let

$$W_i = \hat{W}_i + \lambda_1 I_n \tag{3.11}$$

where λ_1 is greater than the largest eigenvalue of W_i. Note that $-\hat{W}_i \in \mathcal{S}_+^n$ (transformations similar to (3.11) have appeared in the design of optimization algorithms, specifically, in the context of solving the trust region subproblem [12]).[5] Thus

$$\begin{aligned} x^T W_i x &= x^T(\hat{W}_i + \lambda_1 I_n)x \\ &= x^T \hat{W}_i x + \lambda_1 \|x\|^2 \\ &= x^T \hat{W}_i x^T + \lambda_1 \eta^2 \\ &\geq b_i. \end{aligned}$$

Hence, (3.10) is feasible if and only if the set defined by inequalities

$$-x^T \hat{W}_i x \leq \lambda_1 \eta^2 - b_i, \qquad i = 1, \dots, m \tag{3.12}$$

$$x^T x = \eta^2 \tag{3.13}$$

is feasible. Inequality (3.12), in turn, is equivalent to

$$\mu_1 \left(p_1^T x\right)^2 + \cdots + \mu_n \left(p_n^T x\right)^2 \leq \lambda_1 \eta^2 - b_i \tag{3.14}$$

where $-\hat{W}_i = PDP^T$

$$P = [p_1, \dots, p_n]^T \quad (p_i \in \mathbf{R}^n)$$

and

$$D = \mathbf{Diag}([\mu_1, \dots, \mu_n]^T)$$

$(\mu_j \geq 0, j = 1, \dots, n)$. The quadratic inequality (3.14) is convex, assuming an equivalent LMI representation

$$\begin{bmatrix} I_n & l(x) \\ l(x)^T & \lambda_1 \eta^2 - b_i \end{bmatrix} \geq 0 \tag{3.15}$$

with

$$l(x) := \left[\sqrt{\mu_1} p_1^T x, \dots, \sqrt{\mu_n} p_n^T x\right]^T.$$

Thereby, each constraint defining the set $\mathcal{W}$ (3.10)—except the implicit magnitude constraint in (3.9)—can be rewritten in terms of a convex quadratic inequality or a linear matrix inequality; the statement of the proposition thus follows. ∎

We note that the proof of Proposition 3.1 also implies that if the set $\mathcal{B}_\eta$ consists of elements that satisfy the relaxed constraint

$$\|x\| \geq \eta$$

for some real positive number η, then (3.15) will only be a sufficient—and not necessary—condition for guaranteeing the inequality

$$x^T W_i x \geq b_i.$$

Proposition 3.1 states that once the magnitude constraint of the unknown variable is implicitly satisfied, any set of nonconvex quadratic inequalities in this variable can be transformed to a set of LMIs. This metamorphosis does not completely eliminate nonconvexity *per se*, as the relation that restricts the magnitude of a constrained vector is in fact, nonconvex. Nevertheless, as we demonstrate shortly, for the attitude constrained control problem, this possible complication does not arise; the constrained state (the unit quaternion) is always guaranteed to belong to set the $\mathcal{B}_1$ through the kinematic constraint (2.4).

A. Discretization

In this section, we discretize the original Q-CAC problem via Euler's first-order discretization method. The original Q-CAC problem is subsequently transformed to finding the control torques $u_i(k)(i = 1, 2, 3)$ for $k = 0, \dots, N-1$, such that given the initial and terminal conditions

$$(\omega(0), q(0)) \quad (\omega(N), q(N))$$

with

$$\begin{aligned} \omega(k) &= [\omega_1(k), \omega_2(k), \omega_3(k)]^T \\ q(k) &= [q_1(k), q_2(k), q_3(k), q_4(k)]^T \end{aligned}$$

the objective

$$\|\omega(k+1) - \omega(N)\|^2 + \|C(q(k+2) \otimes q(N)^{-1})\|^2 \tag{3.16}$$

is minimized, where

$$C := [I_3 \quad 0_{3\times1}].$$

The constraints of this optimization problem are as follows: The kinematic and dynamic constraints[6]

$$\begin{aligned} F(k)x(k) &= y(k) \\ |G_1 x(k)| &\leq \gamma_1 [1 \quad 1 \quad 1]^T \\ |G_2 x(k)| &\leq \gamma_2 [1 \quad 1 \quad 1]^T \end{aligned}$$

with

$$F(k) = \left[\begin{array}{c|c|c} -I_3 & J & 0_{4\times4} \\ \hline 0_{3\times3} & R(k) & I_4 \end{array}\right]$$

where

$$\begin{aligned} J &:= \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} \\ R(k) &:= \frac{\Delta t}{2} \begin{bmatrix} -q_4(k) & q_3(k) & -q_2(k) \\ -q_3(k) & -q_4(k) & q_1(k) \\ q_2(k) & -q_1(k) & -q_4(k) \\ q_1(k) & q_2(k) & q_3(k) \end{bmatrix} \\ G_1 &:= [I_3 \quad 0_{3\times7}] \\ G_2 &:= [0_{3\times3} \quad I_3 \quad 0_{3\times4}] \\ x(k) &:= \begin{bmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \\ w(k+1) \\ q(k+2) \end{bmatrix} \\ y(k) &:= \begin{bmatrix} J_1 w_1(k) + \Delta t(J_2 - J_3)\omega_2(k)\omega_3(k) \\ J_2 w_2(k) + \Delta t(J_3 - J_1)\omega_3(k)\omega_1(k) \\ J_3 w_3(k) + \Delta t(J_1 - J_2)\omega_1(k)\omega_2(k) \\ w(k) \\ q(k+1) \end{bmatrix} \end{aligned}$$

(the sampling time Δt is assumed to be fixed), and the attitude constraints

$$q(k+2)^T \tilde{A}_i q(k+2) = x(k)^T H_i x(k) \leq 0 \qquad i = 1, \dots, m \tag{3.17}$$

where

$$H_i = \begin{bmatrix} 0_{6\times6} & 0_{6\times4} \\ 0_{4\times6} & \tilde{A}_i \end{bmatrix}$$

and $\tilde{A}_i$ is as defined in (2.6). The second expression in (3.16) is employed in our performance index for the reorientation problem as it

[5]This similarity was pointed out to us by V. Balakrishnan.

[6]The notation $|x|$ refers the vector $[|x|_1, \dots, |x|_n]^T$.

accurately represents the error between the controlled attitude and the desired attitude.[7] In fact

$$q(k)\otimes q(N)^{-1} = q(k)\otimes \begin{bmatrix}-q_1(N)\\-q_2(N)\\-q_3(N)\\q_4(N)\end{bmatrix} = Q(N)q(k)$$

where

$$Q(N) := \begin{bmatrix} q_4(N) & q_3(N) & -q_2(N) & -q_1(N)\\ -q_3(N) & q_4(N) & q_1(N) & -q_2(N)\\ q_2(N) & -q_1(N) & q_4(N) & -q_3(N)\\ q_1(N) & q_2(N) & q_3(N) & q_4(N)\end{bmatrix}.$$

B. Control Torques

For the index set $k = 0, \dots, N-1$, we now consider the quadratically constrained quadratic program

$$\min_{x(k)} \quad \begin{bmatrix}x(k)\\1\end{bmatrix}^T E(k)\begin{bmatrix}x(k)\\1\end{bmatrix} \tag{3.18}$$

subject to

$$F(k)x(k) = y(k)$$
$$|G_1x(k)| \le \gamma_1[1 \quad 1 \quad 1]^T \tag{3.19}$$
$$|G_2x(k)| \le \gamma_2[1 \quad 1 \quad 1]^T \tag{3.20}$$
$$x(k)^TH_ix(k) \le 0, \qquad i = 1,\dots,m \tag{3.21}$$

where

$$E(k) = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times4} & 0_{3\times1}\\ 0_{3\times3} & I_3 & 0_{3\times4} & \omega(N)\\ 0_{4\times3} & 0_{4\times3} & 0_{4\times4} & 0_{4\times1}\\ 0_{1\times3} & \omega(N)^T & 0_{1\times4} & \omega(N)^T\omega(N)\end{bmatrix} + \begin{bmatrix} 0_{6\times6} & 0_{6\times4} & 0_{6\times1}\\ 0_{4\times6} & Q(N)^TQ(N) & 0_{4\times1}\\ 0_{1\times6} & 0_{1\times4} & 0_{1\times1}\end{bmatrix} \in \mathcal{S}_+^{11}$$

$F(k) \in \mathbf{R}^{7\times10}$, $G_1, G_2 \in \mathbf{R}^{10\times10}$, and $H_i \in \mathcal{S}^{10}$. As the *linear* constraints (3.19)–(3.21) guarantee that $\|q(k)\| \approx 1$ at each time index k for small values of Δt, Proposition 3.1 implies that the spacecraft constrained control attitude problem, as augmented with an *arbitrary* number of nonconvex quadratic constraints, can accurately be represented by a (convex) SDP [15]. The convex representation remains valid as long as the errors introduced by linearization are negligible; otherwise, the attitude quaternion is not guaranteed to remain close to a unit vector at each time step. As it pertains to this last remark, we observe that the discretization of (2.4) has the form

$$q(k+1) = \left\{I_4 + \frac{1}{2}\Delta t\Omega(k)\right\}q(k)$$

and, thereby

$$\begin{aligned}\|q(k+1)\|^2 &= q(k)^T\left\{I_4 + \frac{1}{2}\Delta t(\Omega(t)+\Omega(t)^T) + \frac{1}{4}\Delta t^2\Omega(t)^T\Omega(t)\right\}q(k)\\ &= \|q(k)\|^2 + \frac{1}{4}\Delta t^2 q(k)^T\Omega(t)^T\Omega(t)q(k)\\ &\ge \|q(k)\|^2 \end{aligned} \tag{3.23}$$

[7]The notation is adopted from [11].

since

$$\Omega(t) + \Omega(t)^T = 0.$$

The inequality (3.23), along with the initial condition $\|q(0)\| = 1$, implies that

$$\|q(k)\| \ge 1, \qquad \text{for all } k \ge 1. \tag{3.24}$$

By our earlier observation proceeding Proposition 3.1, the SDP approach does still guarantee that the attitude exclusion zone (3.21) is not violated despite errors introduced by the linearization. Nonetheless, as the linearization errors grow, the SDP representation becomes more conservative. The sufficiency of the SDP-based approach to constrained attitude control becomes less restrictive as long as either 1) a proper update rule is employed to propagate the attitude [9], or 2) some positive multiple of the term $\|q(k+2)\|^2$ is included in the objective functional (3.18) that reinforces the desire to keep the magnitude of the attitude updates close to one.

IV. Example

We now revisit our motivational example in Section I where a spacecraft is to collect images of several stars without exposing its on-board sensitive instruments to bright objects in the sky. To simplify our presentation, we assume that only one quadratic constraint—associated with the sun—is present. The physical constants and initial and terminal conditions used in our example are as follows: the spacecraft mass is 1 kg, the inertia matrix along the principle axes is

$$\begin{aligned}J &= \mathrm{Diag}([J_1, J_2, J_3]^T)\\ &= \mathrm{Diag}([100,\ 200,\ 300]^T)\ \mathrm{kg\ m}^2\end{aligned}$$

the initial and the final angular velocities are

$$\omega(t_0) = [0,0,0]^T\ \mathrm{rad/s}$$
$$\omega(t_f) = [0,0,0]^T\ \mathrm{rad/s}$$

the initial and final attitude quaternions are

$$q(t_0) = [0.5000, 0.5000, 0.5000, 0.5000]^T$$
$$q(t_f) = [0.0258, 0.0258, 0.9990, 0.0258]^T$$

the sun vector (in the inertial coordinates) and the camera vector (in the body coordinates) are, respectively

$$x = [0,0,1]^T \quad y = [0.750, 0.433, 0.500]^T$$

and finally, the required angular separation is $\theta = 50$ deg. The initial and the desired final attitude quaternions have been chosen to satisfy (3.17), i.e., the sun is outside the constraint cone emanating from the instrument's bore-sight at the initial and terminal spacecraft configurations. For the purpose of our example, it is convenient to introduce the Q-space consisting of vectors of the form

$$\left\{\left[\frac{q_1}{q_4}, \frac{q_2}{q_4}, \frac{q_3}{q_4}\right]^T |q \text{ a quaternion}\right\}.$$

Fig. 3 depicts the corresponding feasible attitude trajectory in Q-space (solid line) as found via the SDP-based solution strategy (the exclusion zone is represented by the meshed surface extended to infinity). As this figure depicts, the spacecraft attitude evolves along the lower meshed surface—following the boundary of the constraint surface—before reaching the final attitude without violating the sun angle constraint. In our example, the term $\|q(k+2)\|^2$ was included in the objective functional (3.18) at each time step to alleviate the effect of unit magnitude deviation in quaternion updates as introduced by linearization (see Section III-B).

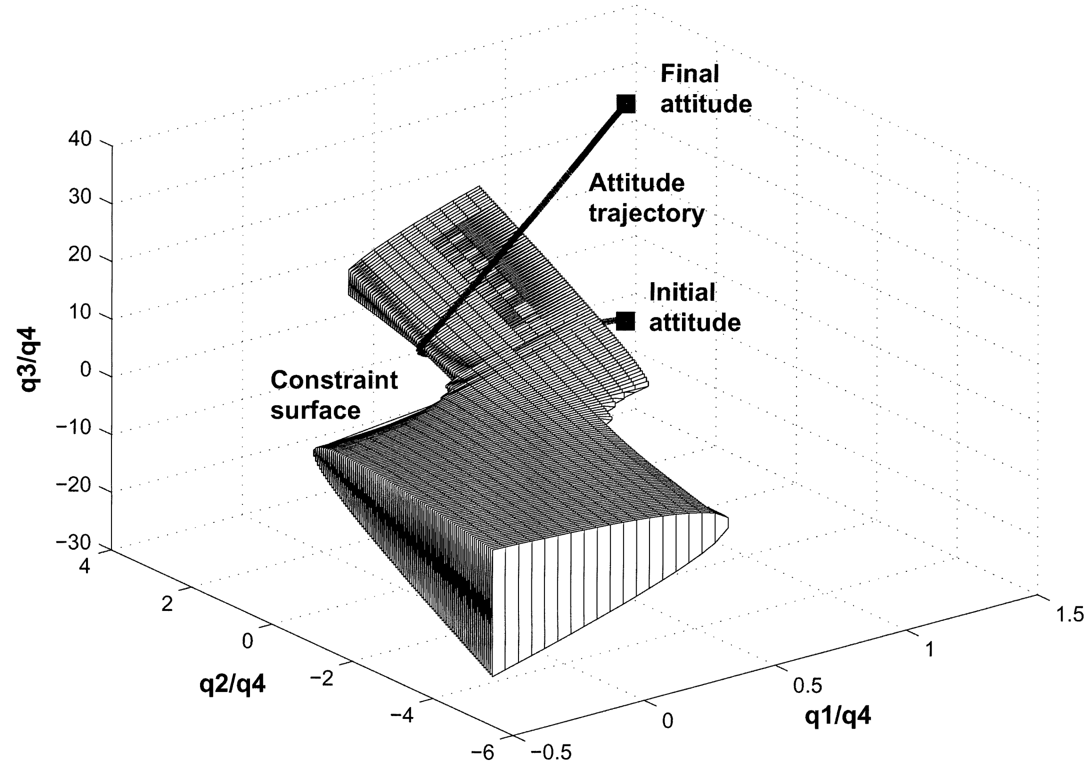

Fig. 3. Attitude trajectory (solid line) in Q-space; the constrained set is the complement of the meshed region.

V. Concluding Remarks

It is generally believed that spacecraft attitude dynamics and control problems are an order of magnitude more challenging than those related to its translational degrees of freedom. In this note, we have explored this idea in the reverse direction, showing that the attitude control problem when augmented with an arbitrary number of nonconvex quadratic constraints on the attitude variable, can in fact be approached via convex optimization. An analogous statement for the translational degrees of freedom is of course not valid. More generally, we note that the approach proposed in this paper is applicable to quadratically state constrained dynamic systems that have an implicitly enforced magnitude constraint on their state vector.

Acknowledgment

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References

[1] A. Ahmed, J. Alexander, D. Boussalis, W. Breckenridge, G. Macala, M. Mesbahi, M. San Martin, G. Singh, and E. Wong, *Cassini Control Analysis Book*. Pasadena, CA: Jet Propulsion Lab., California Inst. Technol., 1998.

[2] A. Ben-Tal and M. Teboulle, "Hidden convexity in some nonconvex quadratically constrained quadratic programming," *Math. Program.*, vol. 72, no. 1, pp. 51–63, 1996.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[4] C. Carrington and J. Junkins, "Optimal nonlinear feedback control for spacecraft attitude maneuvers," *AIAA J. Guid., Control, Dyn.*, vol. 9, no. 1, pp. 99–107, 1986.

[5] E. Gilbert and K. Tan, "Linear systems with state and control constraints: The theory and application of maximal output admissible sets," *Automatica*, vol. 33, no. 12, pp. 2103–2118, 1997.

[6] J. Junkins and J. Turner, "Optimal continuous torque attitude maneuvers," *AIAA J. Guid., Control, Dyna.*, vol. 3, no. 3, pp. 210–217, 1980.

[7] Y. Kim, M. Mesbahi, and F. Y. Hadaegh, "Dual-spacecraft formation flying in deep space: Optimal collision-free reconfigurations," *AIAA J. Guid., Control, Dyna.*, vol. 25, no. 2, pp. 375–379, 2003.

[8] J. B. Kuipers, *Quaternions and Rotation Sequences*. Princeton, NJ: Princeton Univ. Press, 1999.

[9] E. J. Lefferts, F. L. Merkley, and M. D. Shuster, "Kalman filtering for spacecraft attitude determination," *AIAA J. Guid., Control, Dyna.*, vol. 5, no. 5, pp. 417–429, 1982.

[10] D. Mayne and W. Schroeder, "Robust time-optimal control of constrained linear systems," *Automatica*, vol. 33, no. 12, pp. 2103–2118, 1997.

[11] M. D. Shuster, "A survey of attitude representations," *J. Astronaut. Sci.*, vol. 41, no. 4, pp. 439–517, 1993.

[12] R. J. Stern and H. Wolkowicz, "Indefinite trust region subproblems and nonsymmetric eigenvalue perturbations," *SIAM J. Optim.*, vol. 5, no. 2, pp. 286–313, 1995.

[13] J. Strum and S. Zhang, "On cones of nonnegative quadratic functions,", CentER Discussion paper, no. 20 001-26, 2001.

[14] S. Vadali, L. Kraige, and J. Junkins, "New results on the optimal spacecraft attitude maneuver problem," *AIAA J. Guid., Control, Dyna.*, vol. 7, no. 3, pp. 378–380, 1984.

[15] L. Vandenberghe and S. Boyd, "Semidefinite programming," *SIAM Rev.*, vol. 38, no. 1, pp. 49–95, 1996.

[16] J. Wen, "The attitude control problem," *IEEE Trans. Automat. Contr.*, vol. 36, pp. 1148–1161, Oct. 1991.

[17] V. Yakubovich, "Nonconvex optimization problem: The infinite-horizon linear-quadratic control problem with quadratic constraints," *Syst. Control Lett.*, vol. 19, no. 1, pp. 13–22, 1992.