Spectral radius minimization for optimal average consensus and output feedback stabilization

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\textbf{A B S T R A C T}

In this paper, we consider two problems which can be posed as spectral radius minimization problems. Firstly, we consider the fastest average agreement problem on multi-agent networks adopting a linear information exchange protocol. Mathematically, this problem can be cast as finding an optimal $W \in \mathbb{R}^{n \times n}$ such that $x(k + 1) = Wx(k)$, $W \mathbf{1} = \mathbf{1}$, $\mathbf{1}^T W = \mathbf{1}^T$ and $W \in \mathcal{E}(\mathcal{G})$. Here, $x(k) \in \mathbb{R}^n$ is the value possessed by the agents at the $k$th time step, $\mathbf{1} \in \mathbb{R}^n$ is an all-one vector and $\mathcal{E}(\mathcal{G})$ is the set of real matrices in $\mathbb{R}^{n \times n}$ with zeros at the same positions specified by a network graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of agents and $\mathcal{E}$ is the set of communication links between agents. The optimal $W$ is such that the spectral radius $\rho(W - \mathbf{1}\mathbf{1}^T/n)$ is minimized. To this end, we consider two numerical solution schemes: one using the qth-order spectral norm (2-norm) minimization (q-SNM) and the other gradient sampling (GS), inspired by the methods proposed in [Burke, J., Lewis, A., & Overton, M. (2002). Two numerical methods for optimizing matrix stability. \textit{Linear Algebra and its Applications}, 351–352, 117–145; Xiao, L., & Boyd, S. (2004). Fast linear iterations for distributed averaging. \textit{Systems & Control Letters}, 53(1), 65–78]. In this context, we theoretically show that when $\mathcal{E}$ is symmetric, i.e. no information flow from the $i$th to the $j$th agent implies no information flow from the $j$th to the $i$th agent, the solution $W^{q\text{-SNM}}$ from the q-SNM method can be chosen to be symmetric and $W^{q\text{-SNM}}$ is a local minimum of the function $\rho(W - \mathbf{1}\mathbf{1}^T/n)$. Numerically, we show that the q-SNM method performs much better than the GS method when $\mathcal{E}$ is not symmetric. Secondly, we consider the famous static output feedback stabilization problem, which is considered to be a hard problem (some think NP-hard): for a given linear system $(A, B, C)$, find a stabilizing control gain $K$ such that all the real parts of the eigenvalues of $A + BK$ are strictly negative. In spite of its computational complexity, we show numerically that q-SNM successfully yields stabilizing controllers for several benchmark problems with little effort.

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1. Introduction and problem statement

A typical scenario in multi-agent missions is for the agents to agree upon a certain quantity or decision based on their current information. For example, suppose that a team of UAVs (unmanned air vehicles) is tracking a moving object and needs to sense continuously relevant data, e.g. position and heading of the object, and to communicate to them to the team members in order to update the current status of the object. In practice, each UAV is likely to have limited data processing power and therefore the tracking must be done in a decentralized manner. In other words, a UAV is only capable of communicating with a limited number of adjacent UAVs. Therefore, one may easily conclude that the mission heavily hinges on the ramifications of the limited information exchange pattern. Such a decentralized tracking problem that requires each agent (processor) to do iterative weighted average operations in a decentralized manner is called the average consensus problem, and has been studied for numerous applications, e.g. mobile ad-hoc and wireless sensor networks. These applications include consensus with statically or dynamically changing information-exchange topologies (Olfati-Saber & Murray, 2004), high-frequency channel noise (Olfati-Saber & Shamma, 2005), corrupted measurement data (Ren, Beard, & Kingston, 2005), network link failures (Cortes, Martínez, & Bullo, 2006), or state-dependent graph settings (Kim & Mesbahi, 2006).

In this paper, we are particularly interested in the optimal matrix $W \in \mathbb{R}^{n \times n}$ (denoted by $W^*$) such that the following rule
\[ x(k + 1) = Wx(k), \]

\[ x(k + 1) = Wx(k), \]
allows $x(k)$ to converge to $1^T x(0)/n$ with minimum $k^*$ within a prescribed tolerance for every $i \in \{1, 2, \ldots, n\}$. Here, $x(k) = [x_1(k), \ldots, x_n(k)]^T$, $x(k)$ is the value possessed by the $i$th agent at time step $k$ on a network (graph) $G$ with a proper information exchange pattern $\mathcal{E}$, and $k \in \{0, 1, 2, \ldots\}$ is the discrete-time step index. A network (graph) $G$ consists of a set $V$ of nodes (agents) $v_j$ ($i = 1, 2, \ldots, n$) and a set $E$ of edges (communication links) $e_{ij}$ ($i, j = 1, 2, \ldots, n, i \neq j$) with weight $w_{ij}$ on $e_{ij}$. The weighting factor $w_{ij}$ is zero if no communication link exists from $i$ to $j$, and can be any real (not necessarily positive) value otherwise. The position of zero weights (no communication links) defines different information exchange patterns and a set $\delta(\mathcal{E})$.

We note that symmetric $\mathcal{E}$, i.e. $e_{ji} = e_{ij}$ implies $e_{ij} \in \mathcal{E}$, does not necessarily imply symmetric weight $W$. For the sake of the average convergence of the rule (1), i.e.

$$\|W^q - 11/n\| < \epsilon$$

for large positive integer $q$ and sufficiently small positive $\epsilon$, the matrix $W \in \delta(\mathcal{E})$ must have the following properties, as observed in Xiao and Boyd (2004):

$$W = 1, \quad 1^T W = 1^T$$

and $\rho(W - 11/n) \leq \rho(W) < 1$, (3)

where $1 \in \mathbb{R}^n$ denotes an all-one vector and $\rho(W)$ is the spectral radius of matrix $X$. This implies that the optimal $W^*$ is obtained when $\rho(W)$ or $q$ is minimized.

Finding $W^*$ is an old problem, although the structure embedded in $W$ may not be old. It can be translated into finding the most stable discrete-time linear system, or finding the fastest mixing Markov chain (discrete-time stochastic process) when $W$ is non-negative (entry-wise). These areas have been popular research topics in the control community. However, finding $W^*$ or minimizing the spectral radius matrix function $\rho(\cdot)$ is known as a very hard problem in general. This is because $\rho(\cdot)$ is continuous but neither convex nor locally Lipschitz (Overton & Womersley, 1988).

For this reason, there are few works in the literature that directly address the problem in question. In Xiao and Boyd (2004), the authors approach the problem by solving the following program:

$$\mathcal{P}_1 : \min_W \|W\|$$

s.t. $W \in \delta(\mathcal{E}), \quad W = 1, \quad 1^T W = 1^T$ (4)

for a given $\mathcal{E}$. The program $\mathcal{P}_1$ minimizes the largest singular value of $W$, $\sigma(W)$, instead of $\rho(W)$. Thus, the solution $W_s$ to $\mathcal{P}_1$ only guarantees the well-known bound

$$\rho(W^s) \leq \rho(W_s) \leq \bar{\sigma}(W_s),$$

where the gap between $\rho(W^s)$ and $\bar{\sigma}(W_s)$ can be unacceptably large in general. In Burke, Lewis, and Overton (2002), the authors propose the so-called (unconstrained) gradient sampling method to minimize the spectral abscissa $\sigma(\cdot)$ (the largest real part of the eigenvalues). The gradient sampling method is a variation of the reliable steepest descent method. It uses gradient information on the neighbourhood of each iteration point $x$, not just the gradient at the single point $x$. As a result, the gradient sampling method becomes particularly useful when it is applied to minimizing a non-smooth (non-differentiable) function such as $\rho(\cdot)$ (see Burke et al. (2002) and Section 2.2 for details).

In this paper, we further develop the foregoing two ideas for finding $W^*$. In the following Section 2.1, we first introduce the $q$-th order spectral norm minimization method (SNM) to improve the solution $W_s$ to $\mathcal{P}_1$. We then show that $1$-SNM can yield a symmetric solution $W^{(1)}_s$ such that $\bar{\sigma}(W^{(1)}_s) = \rho(W^{(1)}_s)$ if $\mathcal{E}$ is symmetric, and, as a consequence, $W^{(1)}_s$ is a local minimizer of the objective function (4) with any positive integer $q$, and thus of the function $\rho(W)$. In Section 2.2, we propose the constrained gradient sampling method (CGSM) to accommodate non-smooth function minimization problems with general constraints, wherein we provide formula for computing the gradient of $\rho(\cdot)$. We finally compare the two methods by extensive numerical tests in Section 3. These numerical tests suggest that $q$-SNM is a better choice than CGSM when the information exchange pattern $\mathcal{E}$ is non-symmetric.

After the numerical tests for optimal average consensus, we further delineate the $q$-SNM’s superiority in non-symmetric cases in Section 4. We consider the classical static output feedback stabilization problem which has long been considered a hard problem (Syrmos, Abdallah, Dorato, & Grigoriadis, 1997); for a given state model $A, B, C$, find a stabilizing $K$ such that all the eigenvalues of $A + BK$ are strictly negative. We consider several benchmark problems in which all associated $A + BK$ have non-symmetric structures. It is then shown numerically that $q$-SNM successfully yields stabilizing controllers with little effort, in spite of the problems’ notorious reputation. Concluding remarks follow in Section 5.

2. Optimal average consensus and two methods

2.1. q-SNM

The $q$-th order spectral norm minimization method ($q$-SNM) basically solves the following program: for a given $\mathcal{E}$ and a positive integer $q$

$$\mathcal{P}_q^{(q)} : \min_W \|W^q - 11/n\|$$

s.t. $W \in \delta(\mathcal{E}), \quad W = 1, \quad 1^T W = 1^T$, (5)

where $\| \cdot \|$ denotes the 2-norm of a matrix. Note that $\mathcal{P}_q^{(1)} = \mathcal{P}_1$. If there exists $W$ such that (2) is satisfied for some fixed $q$ and $\epsilon$, $W$ must be a feasible solution to $\mathcal{P}_q^{(q)}$. Since we are interested in finding the least $q$, we first solve (4) with $q = 1$ and then increase $q$ to enlarge the feasible set $W_q$, where

$$W_q = \{ W | W = 1, \quad 1^T W = 1^T, \quad W \in \delta(\mathcal{E}), \quad \|W^q - 11/n\| < \epsilon \}$$

for given $\epsilon$ and $\mathcal{E}$ (note that $W_q \subseteq W_{q_2}$ for $1 \leq q_1 < q_2$). In other words, we aim to find the smallest $q$ such that $W_q$ is non-empty.

Unfortunately, $\mathcal{P}_q^{(q)}$ is not convex for $q > 1$ and thus cannot be exactly solved using convex programming techniques. For this reason, we consider approaching the optimal solution to $\mathcal{P}_q^{(q)}$ from a feasible solution by iteratively solving the following dynamic version of $\mathcal{P}_q^{(q)}$, i.e.

$$\min_{W(t)} \|X(t) - 11/n\|^T$$

s.t. $W(t) \in \delta(\mathcal{E}), \quad W(t) = 1, \quad 1^T W(t) = 1^T$, $X(t) = W(t)^q$.
For the sake of removing nonlinearity from the program, we note that (under the assumption that $W(t)$ is differentiable with respect to $t$)

$$X(t) = W(t)^g$$  \hspace{1cm} (5)

if and only if $X(0) = W(0)^g$ and

$$X'(t) = (W(t)^g)' = W'(t)W(t)^{g-1} + W(t)W'(t)W(t)^{g-2} + \ldots + W(t)^{g-1}W'(t).$$

We now discretize the program via $X_0 := X(0), W_0 := W(0), X'(t) := \frac{(X_i - X_{i-1})}{\Delta t}$ and $W'(t) := \frac{(W_i - W_{i-1})}{\Delta t}$, where $\Delta t$ is a small positive constant and $i$ is the time index. Note that the discretization is valid provided that $\|W_i - W_{i-1}\|$ is sufficiently small. In other words, the nonlinear constraints can be treated as linear constraints via differentiation and linearization under certain restrictions. Thus, we actually solve the following $\mathcal{P}_s^{(q)}$ for $W_i$ and $X_i$ instead of $\mathcal{P}_s^{(q)}$: for given $W_{i-1}$ and $X_{i-1} = W_{i-1}^q$,

$$\mathcal{P}_s^{(q)} : \quad \min_{W_i} \|X_i - 11^T/n\|$$

s.t. $W_i \in \delta(\tilde{\gamma}), W_1 = 1, \ 1^T W_i = 1$, $X_i = X_{i-1} + (W_i - W_{i-1}) W_{i-1}^q + W_{i-1}^q (W_i - W_{i-1}) W_{i-1}^q + \ldots + W_{i-1}^q (W_i - W_{i-1})$

and

$$\|W_i - W_{i-1}\| \leq \delta,$$

where $\delta > 0$ is a sufficiently small constant and varies dynamically (see the algorithm $A_s$ below).

The choice of initial matrix $W_0$ is crucial for obtaining a high-quality solution to $\mathcal{P}_s^{(q)}$. Since the program $\mathcal{P}_s^{(q)}$ (before linearization) can be globally solved using a convex program, we use the solution $W_1^q$ to $\mathcal{P}_s^{(q)}$ as the initial condition for $\mathcal{P}_s^{(q)}$. Recalling (2), suppose we fix $\epsilon \in (0, 1)$ and, after solving $\mathcal{P}_s^{(q)}$, we have

$$\|W_1^q - 11^T/n\| \leq \epsilon_1,$$

which implies

$$\|(W_1^q)^g - 11^T/n\| = \|(W_1^q)^{(1)} - 11^T/n\|^g \leq \|W_1^q - 11^T/n\|^g \leq \tilde{\gamma}^g \|W_1^q\|^g \leq \epsilon^g,$$

If $\epsilon^g < \epsilon$, then $W_1^q$ solves $\mathcal{P}_s^{(q)}$. Otherwise, we search for a better solution $\tilde{W}$ near $W_1^q$ in the sense that $\rho(\tilde{W}) < \rho(W_1^q)$. Since the least $\tilde{q}$ such that (2) is satisfied for the fixed $\epsilon$ is unknown, we start with $q = 2$ and thus solve $\mathcal{P}_s^{(q)}$ with $W_{i-1} = W_1^q$ and $X_{i-1} = (W_1^q)^2$, in the hope of finding $\tilde{W}(2)$ such that $\|W(2)^q\|^g < \epsilon$. Once $W_1^q$ is obtained, we check if $\rho(W(2)^q) < \rho(W_1^q)$. If so, we iterate again solving $\mathcal{P}_s^{(q)}$ to improve the solution with new $W_{i-1} = W_1^q$ and $X_{i-1} = (W_2)^q$. Otherwise, we solve the same problem with a smaller $\delta$ in (6) for a refined search or (if $\delta$ is too small) move on to $\mathcal{P}_s^{(q+1)}$ and repeat the same procedure until $\delta$ is sufficiently large (typically in our experience 7). The following summarizes the aforementioned algorithm $A_s$:

**Initialization:** Fix $\delta := 10^{-3}$ and set $q := 2$.

**Step 1:** Solve $\mathcal{P}_s^{(q)}$, obtain $W_1^q$ and set $W^* := W_1^q$, $W_{i-1} := W_1^q$ and $X_{i-1} := W_{i-1}^q$. If $\rho(W_1^q) < \rho(W^*)$, set $W^* := W_1^q$, $W_{i-1} := W_1^q$ and $X_{i-1} := W_{i-1}^q$ and proceed with Step 2; else if $q > 3$, terminate the algorithm; else $\delta := 10^{-5}$, set $q := q + 1, W_{i-1} := W_{i-1}^q$, $\delta := 10^{-3}$ and proceed with Step 2; else $\delta := 0.13$ and proceed with Step 2.

For a fixed $\epsilon$, there is no guarantee that the proposed algorithm finds a solution, $W \in W_*$, and it is a challenge to determine how close the solution $W$ is to the optimal solution $W^*$. However, the following result provides us with a partial answer to this question.

**Proposition 1.** For a network graph having symmetric $\tilde{\gamma}$, $W_1^q$ can be chosen to be symmetric, so that

$$\tilde{\sigma}(W_1^q - 11^T/n) = \rho(W_1^q - 11^T/n).$$

Furthermore, the symmetric $W_1^q$ is a local minimum of the objective functional in (4) with any positive integer $q$.

**Proof.** The first claim easily follows by the convexity of the norm function. In fact, we first note that

$$\gamma W_1^q + (1 - \gamma)(W_1^q)^T$$

for any $\gamma \in [0, 1]$, is a feasible solution to $\mathcal{P}_s^{(q)}$ when $\delta$ is symmetric. In addition,

$$\|W - 11^T/n\| = \gamma \|W - 11^T/n\| + (1 - \gamma)\|W - 11^T/n\| \geq \|\gamma W + (1 - \gamma)W^T - 11^T/n\|.$$

Thus, if $W_1^q$ is a global minimizer of the objective functional of $\mathcal{P}_s^{(q)}$, so is $\gamma W_1^q + (1 - \gamma)(W_1^q)^T$ for any $\gamma \in [0, 1]$. This implies that $\|W_1^q - 11^T/n\| = \|\gamma W_1^q + (1 - \gamma)(W_1^q)^T - 11^T/n\|$. Setting $\gamma = 1/2$, we have another global symmetric minimizer, $(W_1^q + (W_1^q)^T)/2$.

For the second claim, we only consider $q = 2$ because a similar argument can be applied to higher $q$. We first note that for a symmetric matrix $\tilde{V}$,

$$\|\tilde{V}\|^g = \|\tilde{V}\|^g.$$
Suppose \( \text{Horn} \) reveals two facts. The first part of the statement \( \text{P} \) is (and its conjugate pair) is algebraically simple, i.e., has multiplicity motivated by that of Theorem 6.3.12 in \( \text{Horn and Johnson (1985)} \), which allows the eigenvectors to be normalized such that \( (u - i\bar{v})(v + i\bar{v}) = 1 \). If we differentiate the normalized condition with respect to \( x_i \), we have
\[
(u' - i\bar{v}')((v + i\bar{v}) + (u - i\bar{v})'(v' + i\bar{v}')) = 0.
\]
Consider
\[
(u - i\bar{v})\bar{T}(v + i\bar{v}) = \lambda(u - i\bar{v})(v + i\bar{v}) = \lambda
\]
and differentiate the last equality; we then have
\[
\lambda' = ((u' - i\bar{v}')\bar{T}(v + i\bar{v}) + (u - i\bar{v})'(v' + i\bar{v}')) = \lambda((u' - i\bar{v}')\bar{T}(v + i\bar{v}) + (u - i\bar{v})'(v' + i\bar{v}'))
\]
and
\[
\frac{\partial \rho(x)}{\partial x_k} = \Re(\lambda) \frac{u'(X_kv + \bar{u}X_k\bar{v})}{|\lambda|} + \frac{\Im(\lambda)}{|\lambda|} (u'X_k\bar{v} - \bar{u}X_kv).
\]
where \( u + i\bar{v} \) and \( v + i\bar{v} \) are the left and right eigenvectors associated with \( \lambda \), respectively.

**Proof.** The conditions given in the statement guarantee that \((u - i\bar{v})(v + i\bar{v}) \neq 0\) (see Lemma 6.3.10 in \( \text{Horn and Johnson (1985)} \)), which allows the eigenvectors to be normalized such that \((u - i\bar{v})(v + i\bar{v}) = 1\). If we differentiate the normalized condition with respect to \( x_i \), we have
\[
(u' - i\bar{v}')((v + i\bar{v}) + (u - i\bar{v})'(v' + i\bar{v}')) = 0.
\]

For numerical simulations, we use Algorithm 1 proposed in \( \text{Burke et al. (2002)} \), as shown in Table 1, with the following changes: (1) In Step 0, we choose, e.g., by solving a linear program, initial non-zero entries \( x_i \) \( (i = 1, \ldots, m) \) of \( X \) such that \( X_{11} = 1 \), \( 1^T X = 1 \) and \( X \in \delta(\varepsilon) \) for a given network graph \( \varepsilon \). Note that the number of decision variables (non-zero entries of \( X \)) is uniquely defined for each fixed \( \varepsilon \). In order to satisfy the constraints \( 1^T X = 1 \) and \( 1^T Y = 1 \), we need \( 2n \) linear constraints \( h_j(x_i) = 0 \) \( (j = 1, 2, \ldots, 2n) \). Recalling the constrained steepest descent method, we therefore choose a direction \( d \) for each sampled \( Y \) such that
\[
\nabla \rho(Y) d \leq 0,
\]
where \( \nabla \rho(Y) \) is the gradient of \( \rho(\cdot) \) at \( Y = 11^T/n \), and
\[
\nabla h_j(x_1, x_2, \ldots, x_m)^T d = 0 \quad \forall j = 1, 2, \ldots, 2n.
\]
If there are multiple feasible directions \( d \), we choose the one such that \( \nabla \rho(Y) d \) is minimized.

We now present the gradient formula of \( \rho(\cdot) \). The proof is motivated by that of Theorem 6.3.12 in \( \text{Horn and Johnson (1985)} \), and could be deduced from existing works, e.g., \( \text{Burke and Overton (2001)} \).

**Proposition 2.** Suppose \( \rho(\cdot) \) is differentiable at \( X \), and \( \lambda = \Re(\lambda) + i\Im(\lambda) \) is the largest in magnitude eigenvalue of \( X = \sum_i x_i \mathbf{x}_i \) and it (and its conjugate pair) is algebraically simple, i.e., has multiplicity

one. Then, for \( k \in \{1, 2, \ldots, m\} \)
\[
\frac{\partial \rho(X)}{\partial x_k} = \frac{\Re(\lambda)}{|\lambda|} (u'X_kv + \bar{u}X_k\bar{v}) + \frac{\Im(\lambda)}{|\lambda|} (u'X_k\bar{v} - \bar{u}X_kv),
\]
where \( u + i\bar{v} \) and \( v + i\bar{v} \) are the left and right eigenvectors associated with \( \lambda \), respectively.

**3. Numerical tests for optimal average consensus**

In this section, we present test examples to show the efficacy of the proposed two algorithms. We first show how much \( q \)-SNM can improve the solutions obtained via 1-SNM, and then compare \( q \)-SNM and CGSM. For each fixed number \( n \) of agents and the number \( m \) of zeros in the solution matrix \( W \) in (1), we randomly generate fifty symmetric and, respectively, non-symmetric information exchange pattern \( \varepsilon \) and apply the proposed algorithms to each case.
Table 1
Algorithm 1 (Gradient Bundle) proposed in Burke et al. (2002).

0. (initialization) Choose an initial feasible point \( x \in \mathbb{R}^n \), an initial positive value for the sampling radius \( \epsilon \), a positive radius factor \( \theta < 1 \), a positive integer \( N \) defining the number of gradients per bundle, and two positive integers \( M_1 \) and \( M_2 \) to terminate the iterations. Initiate \( k = 1 \).
1. (Inner Iteration) Carry out minimization \( k \) controlled by the sampling radius \( \epsilon \), as follows. Initialize \( j = 0 \).
   (a) Define a bundle \( G \) as the set of \( N \) gradients \( \{y_j \} \), where \( y \) takes on \( N \) values: the current iterate \( x \) and \( N - 1 \) other vectors differing from \( x \) by vectors whose entries are obtained by sampling from a uniform distribution on \([-\epsilon/2, \epsilon/2]\).
   (b) Define the search direction \( d = -\arg\min\{\|v\|_2 : v \in \text{conv } G\} \). If \( d = 0 \), go to Step 2.
   (c) Use a line search to find a positive step-length satisfying \( f(x + td) < f(x) \) with \( t \in (0, \bar{t}] \), where \( \bar{t} = \max\{t: \|x + td\|_\infty \leq \chi\} \) and \( \chi \) is a large constant.
   (d) Replace \( x \) by \( x + td \). If \( t = \bar{t} \), terminate. If \( j < M_2 \), increment \( j \) and return to Step 1. Otherwise, terminate.
2. (Decrease Sampling Parameter) If \( k < M_1 \), replace \( \epsilon \) by the smaller value \( \epsilon/\theta \), increment \( k \) and return to Step 1. Otherwise, terminate.

Table 2
The average fractions of \( \rho(W_1^{(q)}) \) for various \( n \) and \( m \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m )</th>
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<th>4-SNM</th>
<th>5-SNM</th>
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Fig. 1. Spectral radius values and computational times for \((n, m) = (5, 5)\) and non-symmetric \( \epsilon \): solid line for q-SNM and dotted line for CGSM.

3.1. q-SNM versus 1-SNM

As discussed before, q-SNM is nothing but 1-SNM when \( \epsilon \) is symmetric. Therefore, all the simulations below are meant for non-symmetric \( \epsilon \). Table 2 tabulates the simulation results for \((n, m) = (5, 5), (5, 10), (10, 20), (10, 30)\) and \((10, 40)\), respectively. Each value represents the average fraction of \( \rho(W_1^{(q)}) \), i.e. \( \rho(W_1^{(q)}/\rho(W_1^{(1)}) \), after iteratively applying q-SNM with different \( q \in \{2, 3, 4, 5\} \), where \( W_1^{(1)} \) is the solution obtained via q-SNM for each randomly generated pattern. As clearly shown, when \( \epsilon \) is non-symmetric, the solutions obtained via 1-SNM are greatly improved by up to over 80% after successively applying q-SNM.

3.2. q-SNM versus CGSM

Figs. 1–4 show comparison results between q-SNM and CGSM in terms of the spectral radii of the obtained solutions and the associated computational times. Figs. 1 and 2 are for \((n, m) = (5, 5)\), Figs. 3 and 4 for \((n, m) = (10, 20)\). Figs. 1 and 3 are for non-symmetric information exchange patterns \( \epsilon \) and Figs. 2 and 4 for symmetric information exchange patterns \( \epsilon \). The dotted lines correspond to CGSM and the solid lines to q-SNM. As the figures show, q-SNM finds the solutions with a lower spectral radius than CGSM for more than 90% of test cases when \( \epsilon \) is non-symmetric. In contrast, when \( \epsilon \) is symmetric, it is
hard to compare the two methods in general. The two methods show similar performance for \((n, m) = (10, 20)\), but q-SNM is much inferior to CGSM for \((n, m) = (5, 5)\). Regarding the computational complexity of the methods, CGSM pertains to the number of samples for surveying gradient information and the size of the quadratic program for deciding the best direction to move at each iteration, whereas q-SNM mainly concerns the size of the quadratic or semi-definite program for solving (4). Since one can not predict which method requires the least number of mathematical programs to solve to reach a solution for a fixed information exchange pattern, the exact estimation of total computational complexities of the two algorithms may not be obtained easily.

4. Static output feedback stabilization and q-SNM

Having noticed that q-SNM performs well particularly for non-symmetric cases, we further test it for the famous static output feedback stabilization problem (SOFP).\(^2\)

The SOFP is stated as follows: for a given linear system

\[
\dot{x}(t) = Ax(t) + Bu(t),
\]

\[
y(t) = Cx(t),
\]

find a gain \(K\) such that \(u(t) = Ky(t)\) stabilizes the system, i.e. all the real parts of the eigenvalues of \(A + BK\) are strictly negative. This simply stated problem is a famous open problem in control theory (Blondel, Vidyasager, & Willems, 1999). With \(u(t) = Ky(t)\), the closed-loop system becomes

\[
\dot{x}(t) = (A + BK)x(t)
\]
or in discrete form

\[
x(k + 1) = (I + \Delta t(A + BK))x(k)
\]
with \(k = 0, 1, 2, \ldots\) and a sufficiently small \(\Delta t > 0\). Note that \(\rho(I + \Delta t(A + BK)) < 1\) implies all the real parts of the eigenvalues of \(A + BK\) are strictly negative, i.e. max Re(\(A + BK\)) < 0. Therefore, the SOFP can be posed as the minimization of \(\rho(I + \Delta t(A + BK))\) over a set of matrices \(K\).

In order to handle the SOFP, the previously proposed algorithm \(A_s\) can be changed as follows:

**Initialization:** Set \(\delta := 10^{-3}, \Delta t := 10^{-2}, q := 2\) and stabilizing control gain \(K := \Phi\).

**Step 1:** Solve \(\beta_s^{(1)}\) and obtain \((W_s^{(1)}, K_s^{(1)})\).

1. If max Re(\(A + BK_s^{(1)}C\)) < 0, terminate the algorithm with \(K := K_s^{(1)}\).

2. Set \(W^* := W_s^{(1)}, W_{l-1} := W_s^{(1)}\) and \(X_{l-1} := W_s^{(1)}\).

**Step 2:** Solve \(\beta_s^{(q)}\) and obtain \((W_s^{(q)}, K_s^{(q)})\).

1. If max Re(\(A + BK_s^{(q)}C\)) < 0, terminate the algorithm with \(K := K_s^{(q)}\).

2. If \(\rho(W_s^{(q)}) < \rho(W^*)\), set \(W^* := W_s^{(q)}, W_{l-1} := W_s^{(q)}\) and \(X_{l-1} := W_s^{(q)}\), and proceed with Step 2;

3. Else if \(q > 7\), terminate the algorithm;

\[
\text{else } \delta < 10^{-5}.
\]

\[
\text{set } q := q + 1, X_{l-1} := W_s^{(q)}, \delta := 10^{-3} \text{ and proceed with Step 2;}
\]

\[
\text{else } \delta := 0.1\delta \text{ and proceed with Step 2.}
\]

where

\[
\beta_s^{(q)} := \min_{W = A + BK} \|W^q\|
\]

and \((W_s^{(1)}, K_s^{(1)})\) is the solution to \(\beta_s^{(1)}\), \(\beta_s^{(q)}\) is similarly defined as we did in Section 2.1, i.e. the linearized version of \(\beta_s^{(q)}\), and \((W_s^{(q)}, K_s^{(q)})\) is the solution to \(\beta_s^{(q)}\) if \(K = \Phi\) after running the algorithm.

The considered system may have no stabilizing static output feedback controllers. As opposed to the previous version of \(A_s\) in Section 2.1, the current version can be terminated before \(q\) reaches its maximum value of 7, i.e. as soon as a stabilizing controller is found.

We now proceed with the following benchmark systems found in the literature (Blondel, Vidyasager, & Willems, 1999; Vidyasager, 1994):

**Case 1:** \(A = \begin{bmatrix} 1 & 1.05 \\ -1.05 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix};
\)

**Case 2:** \(A = \begin{bmatrix} 1 & 1.05 \\ -1.05 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix};
\)

**Case 3:** \(A = \begin{bmatrix} 1 & 1.05 \\ -1.05 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix};
\)

**Case 4:** \(A = \begin{bmatrix} 0.0482 & -1.0100 \\ 0.0000 & 0.0000 \end{bmatrix}, B = \begin{bmatrix} 0.0000 & 0.0000 \\ 0.0000 & 0.0000 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};
\)

**Case 5:** \(A = \begin{bmatrix} 1.3800 & -0.2077 \\ -0.5814 & -4.2900 \end{bmatrix}, B = \begin{bmatrix} 0.0480 & 4.2730 \\ 0.0000 & 0.0000 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix};
\)

The first case is particularly interesting in that the set of \(K\) which stabilizes the system is very small (see Fig. 5). As depicted in the figure, \(A_s\) starts out with \(K = 0.9995\) and moves along the standard root-locus plot until it just enters the stabilizing zone and finds \(K = -1.0342\). Table 3 summarizes the results for all the cases. Note that stabilizing gains are found in a couple of minutes.

\(^2\) Although the SOFP is known to be theoretically difficult, many practical instances of the SOFP can be solved routinely using publicly available software, e.g. HIFOO (a MATLAB package for fixed-order controller design) available on the internet site: http://www.cs.nyu.edu/overton/software/hifoo/.
and q-SNM is nothing but 1-SNM. We also showed through extensive numerical simulations that q-SNM offers much better performance than CGSM when \( \delta \) is non-symmetric. The numerical simulation result was then elaborated in concert with the famous static output feedback problem, and was further supported by the application of q-SNM to several benchmark systems. Based on the aforementioned results, we believe that the proposed q-SNM method can offer a promising approach to many difficult optimization problems.

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References


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