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A SET-VALUED APPROACH TO OBSERVABILITY*

KHALID KASSARA†

Abstract. We restate observability in a set-valued setting which generalizes the standard output equation, then a close connection to viability kernels is established. Both global and local cases are studied by means of a set of single-valuedness results. Among these, we highlight the results which are derived by monotone set-valued mappings theory and the ones using contingent and paratingent derivatives. Moreover, this new setting gives rise to the notions of maximal observability domains and minimal unobservability domains, which we characterize in the framework of antitone mappings and their fixed points.

Key words. single-valuedness, viability theory, monotone maps, derivatives of set-valued maps, antitone mappings

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1. Introduction and statement of the problem.

Observability means that initial data of a dynamical system can be uniquely determined from the available observation data. For the finite dimensional continuous systems under consideration in this paper, and except for the linear case, most work carried out until now has led to local results; see, for instance, Bartosiewicz [3], Hermann and Krener [13], Isidori [14], Sontag [18], and Sussmann [19]. They use a geometric approach based on the (repeated) Lie derivatives of the output mapping with respect to the system vector fields, locally around a given state.

This paper continues the line of research started in [12] on observability of systems governed by ODEs within a general set-valued context, which can be presented as follows. Let $t_f > 0$ and f be a function from $[0, t_f] \times \mathbb{R}^N$ into $\mathbb{R}^N$ for an integer N. Consider the system

$$\dot{z} = f(t, z),\ t \in [0, t_f]\,, \tag{1.1a}$$

with output given via a subset Θ of $[0, t_f] \times \mathbb{R}^N$, as follows:

$$(t, z(t)) \in \Theta. \tag{1.1b}$$

We call subset Θ the *output domain.* This new output expression stands for the key idea that motivates this study; its benefits can be listed as follows:

(i) It generalizes the standard output equation. In fact, assume the latter is given by

$$\theta(t) = h(t, z(t)),\quad t \in [0, t_f]\,, \tag{1.2a}$$

for given functions θ and h. Then the associated output domain can obviously be expressed as follows:

$$\Theta_s \doteq \{(t, z) \in [0, t_f] \times \mathbb{R}^N \mid \theta(t) = h(t, z)\}. \tag{1.2b}$$

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†Department of Mathematics and Computer Science, University Hassan II, P. O. Box 5366, Maarif 20100, Casablanca, Morocco (kassarak@member.ams.org).

(ii) It may be used to model incomplete or incertain information and even may describe any phenomena involving the state of the system.

(iii) It presents output as a viability condition, allowing use of various technics of viability theory and set-valued analysis.

An appropriate definition of observability can then be adapted to this setting: we say that state $z_0 \in \mathbb{R}^N$ generates output domain Θ on the horizon $[t_0, t_f]$ if constrained system (1.1) has a solution $\bar{z}$ on $[t_0, t_f]$, which satisfies $\bar{z}(t_0) = z_0$. For notation, we will use

$$z_0 \rightsquigarrow \Theta \text{ on } [t_0, t_f].$$

Two states in $\mathbb{R}^N$ are said to be indistinguishable on horizon $[t_0, t_f]$ if both generate output domain Θ. For a subset Σ in $\mathbb{R}^N$, system (1.1) (or the pair (f, Θ)) is said to be Σ-observable on the horizon $[t_0\ t_f]$ if there are no pairs of distinct indistinguishable states on $[t_0\ t_f]$, which both belong to subset Σ. Then subset Θ is called an Σ-*observability domain* for field f. Note that, logically, this definition involves the case when there are no states in Σ that generate the output Θ.

The role of subset Σ is dictated by practical situations: we may sometimes know a priori a given subset to which the unknown initial data must belong. Note this Σ-setting incompasses both global and local observability, as in the former case it suffices to let $\Sigma \doteq \mathbb{R}^N$, and in the latter, $\Sigma \doteq W$, where W stands for a neighborhood of a given state z_0. If that happens, pair (f, Θ) is said to be locally observable on horizon $[t_0, t_f]$ around z_0.

Within the above setting, we are led to the relevant notion of extremal domains that can be intimately associated with the vector field f and the horizon $[t_0, t_f]$. Later, we will see that observable pairs, the standard case (1.2) included of course, can be easily characterized by using these domains.

DEFINITION 1.1. *We call the maximal observability domain (*maxod*) any maximal element of the set of observability domains, ordered by inclusion order. Minimal elements of the set of unobservability domains are called minimal unobservability domains (*minuds*).*

The objective of this paper is to establish new results pertaining to all the previous concepts, demonstrating that the above set-valued framework can be considered as a unified setting for dealing with observability in several situations: linear, nonlinear, global, local, constrained, etc. For that end, we adapt various tools from set-valued analysis and viability theory, as mainly extracted from the books of Aubin [1], Aubin and Frankowska [2] and Rockafellar and Wets [16].

Also, it is worth mentioning that similar set-valued and viability theory tools have been used by Doyen and Rapaport [10] to introduce and design set-valued observers for nonlinear control systems.

The treatment of extremal output domains introduced by Definition 1.1 relies on a background in antitone mappings in partially ordered sets and their fixed point theory; see Carl, Heikkilä, and Dacić [6] and Dacić [8].

Now, we present the fundamental definitions and notation needed for a comprehensive reading of this paper.

For a Euclidean space, the inner product is denoted by $\langle \cdot, \cdot \rangle$, the corresponding norm by $|\cdot|$, and the ball on an element a with radius $r > 0$ by $B_r(a)$. For a linear functional u we set $u^+ \doteq \{y \mid u(y) \geq 0\}$.

Let F stand for a set-valued mapping, we respectively denote its domain and its graph by

$$\operatorname{dom}(F) \doteq \{x \mid F(x) \neq \emptyset\} \text{ and } \operatorname{gph}(F) \doteq \{(x, y) \mid y \in F(x)\}.$$

Also we will use the notation $F^{-1}(y) \doteq \{x \mid y \in F(x)\}$ to define the inverse of mapping F and $\ker(F) = F^{-1}(0)$ its kernel.

We say that a property holds near a point z_0 iff it holds for all points of a neighborhood of z_0.

In the following, we introduce some [2] tangent cones. Let K denote a subset of the Euclidean space $\mathbb{R}^N$ and $x \in K$. The *contingent* cone to subset K at point x is defined by

$$T_K(x) \doteq \left\{ y \in \mathbb{R}^N \mid \liminf_{h \downarrow 0} \frac{d(x+hy, K)}{h} = 0 \right\},$$

where $d(x, K) \doteq \inf_{y \in D} \|y - x\|$ for each $x \in \mathbb{R}^N$. When subset K is given through equalities and inequalities by

$$K \doteq \{x \mid \lambda(x) = 0, \text{ and } \beta_i(x) \geq 0 \text{ for } i = 1, \ldots, q\}$$

for continuous mappings λ (possibly with vectorial values) and β_i, we have [2] the estimate

$$T_K(x) \subset \{y \mid d\lambda(x)y = 0, \text{ and } d\beta_i(x)y \geq 0 \text{ for } i \in I(x)\}$$

for all x in ∂K at which functions λ and β are differentiable, where

$$I(x) \doteq \{i \mid \beta_i(x) = 0\}.$$

The *paratingent* cone is given by

$$P_K(x) \doteq \left\{ y \in \mathbb{R}^N \mid \liminf_{h \downarrow 0, x' \to_K x} \frac{d(x'+hy, K)}{h} = 0 \right\}.$$

We now define two derivatives of a set-valued mapping F at a point $(x, y) \in \mathrm{gph}(F)$. The contingent derivative is the set-valued mapping $DF(x, y)$, introduced through its graph as follows:

$$\mathrm{gph}(DF(x, y)) = T_{\mathrm{gph}(F)}(x, y).$$

In a similar way, the paratingent derivative $PF(x, y)$ is given by

$$\mathrm{gph}(PF(x, y)) = P_{\mathrm{gph}(F)}(x, y). \tag{1.3}$$

The set-valued mapping F is said to be locally injective around x_0 whenever there exists a neighborhood $W \in \mathcal{N}(x_0)$ such that

$$x_1, x_2 \in W, x_1 \neq x_2 \implies F(x_1) \cap F(x_2) = \emptyset.$$

For a field g, the viability kernel of subset K under g on horizon $[0\ t_f]$ corresponds to the set of all initial states $z_0 \in K$ for which the constrained ODE,

$$\dot{z} = g(z), z \in K \text{ and } z(0) = z_0,$$

has a solution on $[0, t_f]$. It is denoted by $\mathrm{viab}_g(K, 0, t_f)$. The viability kernel of subset K under g is $\mathrm{viab}_g(K, \infty)$ and is denoted $\mathrm{viab}_g(K)$. Subset K is said to be viable under field g on horizon $[0, t_f]$ whenever it coincides with its viability kernel on that horizon. See Aubin [1], Bonneuil [4], Frankowska and Quincampoix [11], and Saint-Pierre [17] for this notion, which will play an important role in this study.

The organization of this paper is described as follows. Section 2 is devoted to output domains, providing some situations in which they are observability domains. The aim of section 3 is to investigate the new extremal domains we have introduced in Definition 1.1. In section 4 we clarify the connection between observability and viability theory, while in section 5 necessary and/or sufficient conditions for Σ-observability are demonstrated. Finally in section 6 we provide new results for local observability.

2. Output domains and observability. In this section we are interested in output domains Θ and their role in achieving observability. We thus assume, unless indicated otherwise, that both horizon $[t_0, t_f]$ and subset Σ are fixed. Let $\mathcal{X} \doteq [t_0, t_f] \times \mathbb{R}^N$, and define the following subsets by

$$J_\Theta \doteq \{z \in \Sigma \mid z \rightsquigarrow \Theta \text{ on } [t_0, t_f]\} \; \forall\, \Theta \in 2^{\mathcal{X}} \setminus \{\emptyset\}. \tag{2.1}$$

Denote by $\mathcal{O}$ and $\mathcal{U}$, respectively, the set of Σ-observability domains of f and the set of Σ-unobservability domains of f, i.e.,

$$\mathcal{O} \doteq \{\Theta \mid (f, \Theta) \text{ is } \Sigma\text{-observable}\} \quad \text{and} \quad \mathcal{U} \doteq \mathcal{O}^c.$$

Then we get the following immediate result.

Proposition 2.1.

(i) $\Theta \in \mathcal{O}$ *iff subset* J_Θ *has almost one element.*

(ii) *If* $\Theta \in \mathcal{O}$ *and* $\Theta' \subset \Theta$*, then* $\Theta' \in \mathcal{O}$*.*

(iii) *If* $\Theta \in \mathcal{U}$ *and* $\Theta' \supset \Theta$*, then* $\Theta' \in \mathcal{U}$*.*

Proof. Statement (i) holds by definition of observability, while (ii) and (iii) follow from (i) and the fact that J_Θ is increasingly dependent upon Θ. ▯

Example 2.2. As a simple illustrative example in one dimension, we consider system $\dot{z} = -z^2$ on horizon $[t_0\ t_f]$. Its solution is given by

$$\bar{z}(t, t_0, z_0) = \frac{z_0}{1 + (t - t_0) z_0} \; \forall\, t \in [t_0\ t_f] \text{ and } z_0 \geq 0.$$

By managing this solution along with output domain,

$$\Theta \doteq [t_0\ t_f] \times [\alpha\ \beta] \quad \text{for} \quad \beta > \alpha > 0,$$

it can be readily verified that subset J_Θ of (2.1) is given by

$$J_\Theta = \{z \mid z \leq \beta \text{ and } z(1 - \alpha(t_f - t_0)) \geq \alpha\},$$

and consequently,

$$J_\Theta = \left| \begin{array}{ll} \emptyset & \text{if} \quad 1 \leq \alpha(t_f - t_0), \\ \\ \left[\frac{\alpha}{1 - \alpha(t_f - t_0)} \quad \beta\right] & \text{if} \quad 1 > \alpha(t_f - t_0). \end{array} \right.$$

It follows that pair $(-z^2, \Theta)$ is observable on $[t_0\ t_f]$ iff

$$1 \leq \alpha(t_f - t_0) \text{ or } 1 - \frac{\alpha}{\beta} \leq \alpha(t_f - t_0) < 1. \tag{2.2}$$

The statements of Proposition 2.1 lead us to set the following noteworthy facts:

(a) According to statement (i), emptyset belongs to $\mathcal{O}$ and in general $\mathcal{X}$ is an element of $\mathcal{U}$.

(b) It follows from (ii) that the intersection is a stable operation for observability domains. On the contrary, note that the union is not. A counterexample actually can be constructed on the basis of characterizing condition (2.2). For $t_f = t_0 + 1$, one can see that output domains $[t_0\ t_f] \times [0.5\ 1]$ and $[t_0\ t_f] \times [1\ 1.5]$ are observability domains of the field $f(t, z) \doteq -z^2$, but domain $[t_0\ t_f] \times [0.5\ 1.5]$ is not.

(c) Suppose that Θ is an open output domain, on which function f is continuous; then Peano's theorem [1, Theorem 1.2.2] can be invoked to examine system (1.1). Let $z_0 \in \Theta_{t_0}$, where

$$\Theta_{t_0} \doteq \{z \mid (t_0, z) \in \Theta\}.$$

Then denote by $[t_0\ \tau_\Theta(z_0))$ the maximal interval (possibly of infinite length) on which the system has a solution $\bar{z}$ issued from z_0 at instant t_0 and satisfies

$$(t, \bar{z}(t)) \in \Theta \ \forall\, t \in [t_0\ \tau_\Theta(z_0)).$$

As a result, subset J_Θ of (2.1) can obviously be expressed by

$$J_\Theta = \{z \in \Sigma \cap \Theta_{t_0} \mid \tau_\Theta(z) > t_f\}.$$

In particular, if $\tau_\Theta(z_0) > t_f$ for at least two distinct z_0's, then Θ is an unobservability domain. That especially arises when subset J_Θ has a nonempty interior, beeing then infinite as it includes a ball. On the contrary, if $\tau_\Theta(z) \le t_f$ for all z in $\Sigma \cap \Theta_{t_0}$, then $J_\Theta = \emptyset$, and hence Θ is an observability domain.

(d) Statement (iii) sets us in a topological context of nets. Actually, when it is endowed with order $\supset$, collection of unobservability domains $\mathcal{U}$ stands for a directed set:

$$\Theta_1 \in \mathcal{U} \text{ and } \Theta_2 \in \mathcal{U} \implies \Theta_1 \cup \Theta_2 \in \mathcal{U}.$$

This motivates us to investigate when subset Σ is compact, taking the opportunity to use the topology notion of cluster point.

THEOREM 2.3. *Assume that Σ is compact; then there exists a subset $\Omega \subset \Sigma$ which satisfies the following statement:*

$$\Omega \cap \operatorname{int} \bigcap_{\substack{\Theta' \in \mathcal{U} \\ \Theta' \supset \Theta}} J^c_{\Theta'} \neq \emptyset \implies \Theta \in \mathcal{O}. \tag{2.3}$$

Proof. Let Ω be the set of cluster points of nets $(x_{\Theta'})_{\Theta' \in \mathcal{U}}$ of subset Σ, satisfying

$$x_{\Theta'} \in J_{\Theta'} \ \forall\, \Theta' \in \mathcal{U}. \tag{2.4}$$

Such nets actually do exist because $J_{\Theta'} \neq \emptyset$ for all $\Theta' \in \mathcal{U}$. Also note that subset Ω is nonempty due to the fact that every net of a compact subset has a cluster point; see [5, Theorem 13.13].

Let Θ be as in the left side of (2.3), and assume that it belongs to collection $\mathcal{U}$. Consider the open subset

$$W \doteq \operatorname{int} \bigcap_{\substack{\Theta' \in \mathcal{U} \\ \Theta' \supset \Theta}} J^c_{\Theta'},$$

and let c belong to $\Omega \cap W$. As c is a cluster point of some net $(x_{\Theta'})_{\Theta' \in \mathcal{U}}$, there exists a subset Θ'_0 in $\mathcal{U}$ such that $\Theta'_0 \supset \Theta$ and $x_{\Theta'_0} \in W$. This is in contradiction with the fact that $x_{\Theta'_0} \in J_{\Theta'_0}$. ☐

3. Maxods and minuds.

3.1. Characterizations.

According to Definition 1.1, a subset is a *maxod* if it is a maximal element of collection of observability domains $\mathcal{O}$. It is a *minud* if it is a minimal element of $\mathcal{U}$. Here both $\mathcal{O}$ and $\mathcal{U}$ are viewed as subsets of the power set $2^{\mathcal{X}}$, ordered by inclusion.

In other words, $\Theta_\star$ is a maxod iff the only output domain Θ satisfying $\Theta \supset \Theta_\star$ and (f, Θ) observable is $\Theta = \Theta_\star$. Likewise, $\Theta^\star$ is a minud iff the only domain Θ such that $\Theta \subset \Theta^\star$ and (f, Θ) is unobservable, is $\Theta = \Theta^\star$.

Out of statement (ii) of Proposition 2.1, a second characterization of maxods and minuds can be provided as follows.

PROPOSITION 3.1. *A maxod $\Theta_\star$, and a minud $\Theta^\star$ are also characterized by the following statements:*

$$\Theta \subset \Theta_\star \implies \Theta \in \mathcal{O} \quad \textit{and} \quad \Theta \supsetneq \Theta_\star \implies \Theta \in \mathcal{U} \tag{3.1a}$$

and

$$\Theta \supset \Theta^\star \implies \Theta \in \mathcal{U} \quad \textit{and} \quad \Theta \subsetneq \Theta^\star \implies \Theta \in \mathcal{O}. \tag{3.1b}$$

As a consequence of Proposition 3.1, involving the relationship between maxods, and the one between minuds, we have the following result.

COROLLARY 3.2. *Any two distinct maxods and any two distinct minuds are necessarily uncomparable.*

Proof. Assume two distinct maxods $\Theta^1_\star$ and $\Theta^2_\star$. Due to expression (3.1b), whenever they were comparable, then one of them would be both observable and unobservable, yielding a contradiction. ☐

It is of interest to note that implications (3.1) present maxods and minuds as subsets separating observability domains from unobservability domains. This separation principle turns out to be easy to use, although only output domains which are comparable are of concern. For instance, as regards standard output equation (1.2), assertions given by (3.1) apply as follows.

COROLLARY 3.3. *Let $\Theta_\star$ be a maxod; then*

$$\begin{matrix} \theta(t) = h(t,x)\ \forall (t,x) \in \Gamma \\ \textit{(with } \Gamma \supsetneq \Theta_\star\textit{)} \end{matrix} \implies (f, (1.2\text{a}))\ \textit{unobservable} \tag{3.2a}$$

and

$$\theta(t) \neq h(t,x)\ \forall (t,x) \in \Theta^c_\star \implies (f, (1.2\text{a}))\ \textit{observable.} \tag{3.2b}$$

Proof. Let Θ_s be as given by (1.2b). Note that the left-hand sides of (3.2a) and (3.2b), respectively, express that $\Theta_s \supsetneq \Theta_\star$ and $\Theta_s \subset \Theta_\star$ when applying assertions given by (3.1). ☐

Define the mappings $\varphi, \psi : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$ as follows:

$$\varphi(\Theta) \doteq \{(s,y) \in \mathcal{X} \mid \Theta \cup \{(s,y)\} \in \mathcal{O}\} \tag{3.3a}$$

and

$$\psi(\Theta) \doteq \{(s,y) \in \mathcal{X} \mid \Theta \setminus \{(s,y)\} \in \mathcal{O}\}. \tag{3.3b}$$

Below, we list some properties satisfied by mappings φ and ψ and which are needed in what follows.

PROPOSITION 3.4. *We have the following statements:*

(i) *Both* φ *and* ψ *are decreasing (with respect to inclusion).*

(ii) $\varphi(\Theta) \subset \psi(\Theta)$ *for all* $\Theta \subset \mathcal{X}$.

(iii) $\Theta \in \mathcal{O}$ *implies* $\Theta \subset \varphi(\Theta)$.

(iv) $\Theta \in \mathcal{U}$ *implies* $\psi(\Theta) \subset \Theta$.

Proof. To show assertion (i), let $\Theta_1 \subset \Theta_2$ be two subsets of $\mathcal{X}$; then by statement (ii) of Proposition 2.1, we get

$$(s,y) \in \varphi(\Theta_2) \implies \Theta_2 \cup \{(s,y)\} \in \mathcal{O} \implies \Theta_1 \cup \{(s,y)\} \in \mathcal{O} \implies (s,y) \in \varphi(\Theta_1).$$

In the same way, for mapping ψ, we get

$$(s,y) \in \psi(\Theta_2) \implies \Theta_2 \setminus \{(s,y)\} \in \mathcal{O} \implies \Theta_1 \setminus \{(s,y)\} \in \mathcal{O} \implies (s,y) \in \psi(\Theta_1).$$

Next we prove assertion (ii). Let Θ be a subset of $\mathcal{X}$; then we have

$$(s,y) \in \varphi(\Theta) \implies \Theta \cup \{(s,y)\} \in \mathcal{O} \implies \Theta \setminus \{(s,y)\} \in \mathcal{O} \implies (s,y) \in \psi(\Theta).$$

To show (iii), suppose that pair (f,Θ) is observable, and let $(s,y) \in \Theta$; then $\Theta \cup \{(s,y)\} = \Theta$, and thereby (s,y) belongs to $\varphi(\Theta)$. Therefore $\Theta \subset \varphi(\Theta)$.

To show (iv), let (s,y) belong to $\psi(\Theta)$. If $(s,y) \notin \Theta$, then $\Theta = \Theta \setminus \{(s,y)\}$ would belong to $\mathcal{O}$. ☐

In the next result we prove a further characterization of maxods and minuds in terms of fixed points.

THEOREM 3.5. *Let* $\mathcal{M}_\star$ *and* $\mathcal{M}^\star$ *denote respectively the set of maxods and the set of minuds; then we have the following statements:*

$$\mathcal{M}_\star = \mathrm{fix}(\varphi) \text{ and } \mathcal{M}^\star = \mathrm{fix}(\psi). \tag{3.4}$$

Proof. Suppose that $\Theta_\star$ is a maxod; then by using (3.1a) we get

$$(s,y) \in \Theta_\star^c \implies (f, \Theta_\star \cup \{(s,y)\}) \text{ is unobservable.}$$

That implies that subset $\varphi(\Theta_\star)$ is included in subset $\Theta_\star$. Since pair $(f,\Theta_\star)$ is observable, then statement (iii) of Proposition 3.4 yields $\Theta_\star \subset \varphi(\Theta_\star)$. Thus, we get $\Theta_\star = \varphi(\Theta_\star)$.

Conversely, assume that $\Theta_\star = \varphi(\Theta_\star)$. To prove that $\Theta_\star$ is a maxod, we begin by checking observability of pair $(f,\Theta_\star)$. Since $\varphi(\Theta_\star) \neq \emptyset$ (as $\varphi(\emptyset) \neq \emptyset$), let (s_0,y_0) belong to $\varphi(\Theta_\star)$; then $\Theta_\star \cup \{(s_0,y_0)\}$ belongs to $\mathcal{O}$ and so does $\Theta_\star$.

Now, let Θ be a subset which includes $\Theta_\star$ and for which pair (f,Θ) is observable. Let $(s,y) \in \Theta$; then $\Theta_\star \cup \{(s,y)\} \subset \Theta$. As a result $\Theta_\star \cup \{(s,y)\} \in \mathcal{O}$ and so $(s,y) \in \varphi(\Theta_\star)$. Therefore $\Theta = \Theta_\star$, and hence $\Theta_\star$ is a maxod.

Next, we proceed to show that $\mathcal{M}^\star = \mathrm{fix}(\psi)$. Assume that $\Theta^\star$ is a minud; then by using (3.1b) we get

$$(s,y) \in \Theta^\star \implies \Theta^\star \setminus \{(s,y)\} \subsetneq \Theta^\star \implies (f, \Theta^\star \setminus \{(s,y)\}) \text{ is observable.}$$

It follows that $\Theta^\star \subset \psi(\Theta^\star)$.

Let (s,y) belong to $\psi(\Theta^\star)$; then $\Theta^\star \setminus \{(s,y)\}$ belongs to $\mathcal{O}$. If $(s,y) \notin \Theta^\star$, it follows that $\Theta^\star \setminus \{(s,y)\} = \Theta^\star$, yielding a contradiction with the fact that $\Theta^\star \in \mathcal{U}$. Consequently, we get $\Theta^\star = \psi(\Theta^\star)$.

Conversely, assume that $\Theta^\star = \psi(\Theta^\star)$. To show that $\Theta^\star$ is a minud, we first check whether it is an unobservability domain. Assume that $\Theta^\star \subsetneq \mathcal{X}$ (otherwise $\Theta^\star = \mathcal{X} \in \mathcal{U}$), and let $(s, y) \notin \Theta^\star$. If $\Theta^\star \in \mathcal{O}$, we would have $(s, y) \in \psi(\Theta^\star)$, yielding a contradiction with $\Theta^\star = \psi(\Theta^\star)$.

Let Θ belong to $\mathcal{U}$ and be such that $\Theta \subset \Theta^\star$. Assume that $\Theta^\star \setminus \Theta \neq \emptyset$ and let (s_0, y_0) belong to $\Theta^\star \setminus \Theta$; then we get $\Theta \subset \Theta^\star \setminus \{(s_0, y_0)\}$, leading to a contradiction originated from the facts that $(s_0, y_0) \in \psi(\Theta^\star)$ and Θ belongs to $\mathcal{U}$. We therefore have proved that $\Theta = \Theta^\star$. ▯

Remark 3.6. Thanks to Theorem 3.5, a nonempty subset Θ is a maxod iff the following condensed statement holds:

$$(s, y) \in \Theta \iff \Theta \cup \{(s, y)\} \in \mathcal{O}.$$

Also, subset Θ is a minud iff

$$(s, y) \in \Theta \iff \Theta \setminus \{(s, y)\} \in \mathcal{O}.$$

3.2. Existence and uniqueness. For the sake of exposition we shall restrict attention to studying maxods, as similar results for minuds can be derived by duality. As a consequence of Theorem 3.5, one is led to seek fixed points of the antitone (i.e., decreasing) self-mapping φ, on the power set $2^{\mathcal{X}}$, equipped with the inclusion order. Define the subset

$$\Gamma \doteq \bigcup_{\Theta \in \mathcal{O}} \Theta, \tag{3.5a}$$

and

$$\Gamma_0 \doteq \bigcap_{\Theta \in \mathcal{O}'} \Theta, \quad \text{where } \mathcal{O}' \doteq \{\Theta \subset \mathcal{X} \mid \varphi(\Theta) \subset \Theta\}. \tag{3.5b}$$

Then we can show the result below.

Theorem 3.7. *The following statements hold:*

(i) *If* $\Gamma \in \mathcal{O}$, *then* Γ *is the unique maxod of* f.

(ii) *If* $\Gamma_0 \in \mathcal{U}$, *then there is no maxod.*

Proof. The hypothesis of the theorem entails existence and nonemptiness of subset Γ. Assume that $\Gamma \in \mathcal{O}$; then by Proposition 3.4(iii), we get $\Gamma \subset \varphi(\Gamma)$. Let $(s, y) \in \varphi(\Gamma)$; then $\Gamma \cup \{(s, y)\} \in \mathcal{O}$, and (3.5a) yields

$$\Gamma \cup \{(s, y)\} \subset \Gamma.$$

This implies that $(s, y) \in \Gamma$, and thus $\varphi(\Gamma) \subset \Gamma$. As a result $\varphi(\Gamma) = \Gamma$, from whence Γ is a maxod. To get its uniqueness, if subset Θ is a maxod, then Θ belongs to $\mathcal{O}$. Thereby $\Theta \subset \Gamma$, and because two distinct maxods must be uncomparable, we deduce that $\Theta = \Gamma$.

As regards assertion (ii), whenever subset Θ is a maxod, it belongs to $\mathcal{O} \cap \mathcal{O}'$ and hence it satisfies $\Gamma_0 \subset \Theta$, yielding a contradiction with unobservability of pair (f, Γ_0). ▯

For computational reasons, it is of interest to connect with fixed points of mapping $\varphi^2 \doteq \varphi \circ \varphi$, the latter being an isotone mapping (i.e., preserving the inclusion order), whose fixed points are dictated by the famous Tarski [6, 7, 8] theorem. This provides the largest fixed point Γ_+ and the least fixed point Γ_-, as follows:

$$\Gamma_+ \doteq \bigcup_{\Theta\subset\varphi^2(\Theta)} \Theta, \quad \text{and} \quad \Gamma_- \doteq \bigcap_{\Theta\supset\varphi^2(\Theta)} \Theta. \tag{3.6}$$

We next show assertions involving the remarkable domains introduced by formulas (3.6).

THEOREM 3.8. *We have the following statements:*

(i) *If* Θ *is a maxod, then* $\Gamma_- \subset \Theta \subset \Gamma_+$.

(ii) *If one of the conditions*

  (a) (f, Γ_+) *is observable,*

  (b) $(s, y) \in \Gamma_-^c$ *implies* $(f, \Gamma_- \cup \{(s, y)\})$ *unobservable*

is satisfied, then Γ_+ *and* Γ_- *do coincide with the unique maxod.*

Proof. Statement (i) holds thanks to Theorem 3.5 and the fact that

$$\text{fix}(\varphi) \subset \text{fix}(\varphi^2).$$

As regards assertion (ii), assume that condition (a) is fulfilled; then $\Gamma_+ \subset \varphi(\Gamma_+)$. Since $\varphi(\Gamma_+)$ is also a fixed point of mapping φ^2, then $\Gamma_+ \supset \varphi(\Gamma_+)$. It follows that $\Gamma_+ = \varphi(\Gamma_+)$, and thereby it is a maxod. Suppose that $\Theta \in \text{fix}(\varphi^2)$; then $\Theta \subset \Gamma_+$. As a result $\varphi(\Theta) \supset \varphi(\Gamma_+)$, that is, $\varphi(\Theta) \supset \Gamma_+$. But, as $\varphi(\Theta)$ is also a fixed point of φ^2, we have $\varphi(\Theta) \subset \Gamma_+$. It follows that $\varphi(\Theta) = \Gamma_+$, and thereby,

$$\Theta = \varphi^2(\Theta) = \varphi(\Gamma_+) = \Gamma_+.$$

We have thus proved that mapping φ^2 has a unique fixed point. That implies that

$$\Gamma_+ = \Gamma_-.$$

If condition (b) is rather assumed, then it is equivalent to $\Gamma_- \supset \varphi(\Gamma_-)$. Since Γ_- is the least fixed point of φ^2, and $\varphi(\Gamma_-)$ is also a fixed point of mapping φ^2, then $\Gamma_- \subset \varphi(\Gamma_-)$. It follows that $\Gamma_- = \varphi(\Gamma_-)$, and thereby it is a maxod. To get its uniqueness as a fixed point of mapping φ^2, the proof is analogous to case (a). ☐

4. Connection with viability theory. In this section, we cover some of the aspects relating observability with viability theory [1]. We introduce the following multifunction by

$$J(t) \doteq \{z \mid z \rightsquigarrow \Theta \text{ on } [t, t_f]\}, \quad t \in [0, t_f)\,. \tag{4.1}$$

As an immediate result, we have the following.

PROPOSITION 4.1. *Let* t_0 *belong to* $[0, t_f)$, $s_f \geq t_f$, *and let* Σ, Σ_1, *and* Σ_2 *be subsets of* $\mathbb{R}^N$ *such that* $\Sigma_1 \subset \Sigma_2$. *Then, we have the following statements:*

(i) *Pair* (f, Θ) *is* Σ*-observable on* $[t_0, t_f]$ *iff subset* $\Sigma \cap J(t_0)$ *has almost one element.*

(ii) *If* (f, Θ) *is* Σ_2*-observable on* $[t_0, t_f]$*, then* (f, Θ) *is* Σ_1*-observable on* $[t_0, t_f]$.

(iii) *If* (f, Θ) *is observable on* $[t_0, t_f]$*, then* (f, Θ) *is observable on* $[t_0, s_f]$.

By (i) above, it can be seen that all the starting times for which the system is observable can be provided by examining the geometry of subset *gph*(J), as illustrated by Figure 4.1. In what follows, we prove an estimate for subset *gph*(J) by means of viability kernels.

Let $g \doteq (1, f)$; then we show the main result of this section.

THEOREM 4.2. *We have the following statement:*

$$\text{viab}_g(\Theta, 0, t_f) \subset \text{gph}(J) \subset \bigcup_{0<\tau<t_f} \text{viab}_g(\Theta, 0, \tau). \tag{4.2}$$

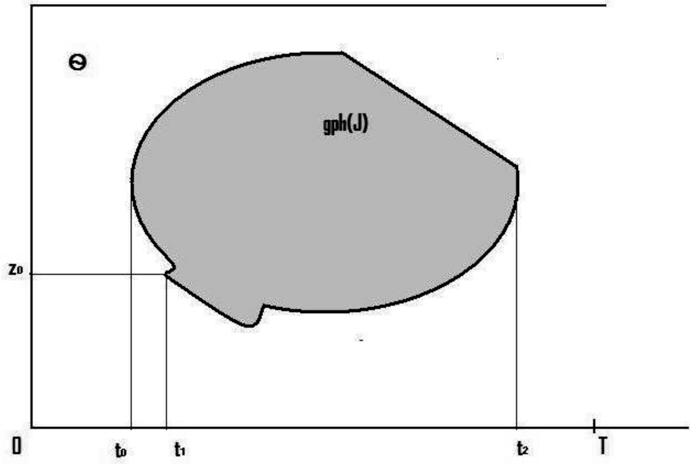

FIG. 4.1. *Observability holds when the vertical section of* gph(J) *has almost one element. In this instance, it holds only on horizons* $[t\ T]$ *for* $t \leq t_0$ *or* $t \geq t_2$*, whereas the property is only local near* z_0*, on horizon* $[t_1\ T]$.

Proof. Let $(t_0, z_0) \in \text{viab}_g(\Theta, 0, t_f)$; then there exists a solution $(\bar{t}, \bar{z})$ satisfying

$$\dot{\bar{t}}(s) = 1, \dot{\bar{z}}(s) = f(\bar{t}(s), \bar{z}(s));\ (\bar{t}(s), \bar{z}(s)) \in \Theta\ \forall s \in [0, t_f]\,, \\ \text{and } (\bar{t}(0), \bar{z}(0)) = (t_0, z_0).$$

Note $\bar{t}(s) = t_0 + s$ for all $s \in [0, t_f]$. Let $\bar{y}(t) = \bar{z}(t - t_0)$ for all $t \in [t_0, t_f]$. Then,

$$\dot{\bar{y}}(t) = f(t, \bar{y}(t)); (t, \bar{y}(t)) \in \Theta\ \forall\ t \in [t_0, t_f] \text{ and } \bar{y}(t_0) = z_0.$$

Therefore $z_0 \in J(t_0)$. Now for $(t_0, z_0) \in \text{gph}(J)$, by going up, we get a viable solution on horizon $[0, t_f - t_0]$. As a result $(t_0, z_0) \in \text{viab}_g(\Theta, 0, \tau)$ with $\tau = t_f - t_0$. □

COROLLARY 4.3. *For pair* (f, Θ) *to be* Σ*-observable on* $[t_0, t_f]$*, it is necessary that it satisfies the statement*

$$\begin{array}{ccc} K \subset \Theta,\ K \textit{ viable} & \Longrightarrow & \textit{at most one } z_0 \in \Sigma \\ \textit{under } g \textit{ on } [0, t_f] & & \textit{is such that } (t_0, z_0) \in K. \end{array} \tag{4.3}$$

Proof. Suppose that (f, Θ) is Σ-observable on $[t_0, t_f]$. Let K as in (4.3) and $(t_0, z_1), (t_0, z_2)$ belong to K with z_i two distincts elements of Σ. Hence there exist two solutions $(\bar{t}_i, \bar{z}_i), i = 1, 2$, satisfying

$$\dot{\bar{t}}_i(s) = 1, \dot{\bar{z}}_i(s) = f(\bar{t}_i(s), \bar{z}_i(s));\ (\bar{t}_i(s), \bar{z}_i(s)) \in K\ \forall s \in [0, t_f]\,, \\ \text{and } (\bar{t}_i(0), \bar{z}_i(0)) = (t_0, z_i).$$

As in the proof of Theorem 4.2, there exist two solutions $\bar{y}_i, i = 1, 2$, which satisfy

$$\dot{\bar{y}}_i(t) = f(t, \bar{y}_i(t)); (t, \bar{y}_i(t)) \in K\ \forall\ t \in [t_0, t_f] \text{ and } \bar{y}_i(t_0) = z_i.$$

Since $K \subset \Theta$, then both z_1 and z_2 belong to $\Sigma \cap J(t_0)$. This contradicts Σ-observability of pair (f, Θ) on $[t_0, t_f]$. □

Similarly, by using the upper estimate in (4.2), we get the following sufficient condition.

Corollary 4.4. *Assume that the following statement holds:*

$$\begin{array}{c}\textit{at most one } z_0 \in \Sigma \textit{ is such that there exists}\\ \tau \in (0\ t_f) \textit{ such that } (t_0, z_0) \in \mathrm{viab}_g(\Theta, 0, \tau).\end{array}$$

Then pair (f, Θ) *is* Σ*-observable on* $[t_0, t_f]$.

We now consider the following condition:

$$\mathrm{viab}_g(\Theta, 0, \tau) \subset \mathrm{viab}_g(\Theta, 0, t_f)\ \forall\, \tau \in (0, t_f)\,. \tag{4.4}$$

Corollary 4.5. *Under condition* (4.4)*, we have*

$$\mathrm{gph}(J) = \mathrm{viab}_g(\Theta, 0, t_f), \tag{4.5}$$

and condition (4.3) *is sufficient for* (f, Θ) *to be* Σ*-observable on* $[t_0, t_f]$.

Proof. The statement of (4.5) is due to both (4.2) and (4.4). Therefore gph(J) is a viable subset of Θ. Let $K \doteq \mathrm{gph}(J)$; then there exists at most one $z_0 \in \Sigma$ such that $(t_0, z_0) \in \mathrm{gph}(J)$. This implies that subset $(\Sigma \cap J(t_0)$ has almost one element. ☐

We now provide some comments about the results above:

(i) Condition (4.4) means that any viable solution on $[0, \tau]$ can be extended to a viable solution on $[0, t_f]$. In particular, it holds if f has linear growth or when output domain Θ is bounded.

(ii) Note that condition (4.4) does not require knowledge of the viability kernel of Θ. It can be checked as follows:

$$\begin{array}{l}K \subset \Theta,\ K \text{ loc. compact and}\\ (1, f(t,z)) \in T_K(t,z)\ \forall (t,z) \in K\end{array} \Longrightarrow \begin{array}{l}\text{at most one } z_0 \in \Sigma\\ \text{is such that } (t_0, z_0) \in K.\end{array}$$

Nonetheless if the viability kernel is available, then it suffices to check whether the subset

$$\{z_0 \in \Sigma \mid (t_0, z_0) \in \mathrm{viab}_g(\Theta, 0, t_f)\}$$

has almost one element.

5. Σ-observability.

We devote this section to the study of Σ-observable pairs on horizons $[t_0\ t_f]$ with $t_0 \in [0\ t_f)$, nonfixed, and for a given susbet Σ in $\mathbb{R}^N$. The single-valuedness results we shall use have relevance to monotone maps theory, convexity, and the contingent derivative.

5.1. Using monotone maps.

Let $\pi : \mathbb{R}^N \to [0\ t_f)$; then consider the statement

$$\begin{array}{l} z_1 \rightsquigarrow \Theta \text{ on } [\pi(y_1)\ t_f]\\ z_2 \rightsquigarrow \Theta \text{ on } [\pi(y_2)\ t_f]\end{array} \Longrightarrow \langle z_2 - z_1, y_2 - y_1\rangle \geq \xi(z_2 - z_1, y_2 - y_1) \tag{5.1}$$

for all z_1, z_2 in Σ and y_1, y_2 in $\mathbb{R}^N$, where function ξ satisfies

$$\begin{array}{c}\text{there exists } \sigma > 0 \text{ such that}\\ \xi(u, u + \sigma v) - |u|^2 \geq 0\ \forall\, u, v \in \mathbb{R}^N.\end{array} \tag{5.2}$$

Then we are ready to show the following result.

THEOREM 5.1. *Suppose that statements* (5.1) *and* (5.2) *hold. Then pair* (f, Θ) *is* Σ*-observable on* $[t_0\ t_f]$ *for all* $t_0 \in \pi(\mathbb{R}^N)$.

Proof. Let G be defined by $G(y) \doteq \Sigma \cap J(\pi(y))$ for all $y \in \mathbb{R}^N$ with J as in (4.1), and let the map F be as follows:

$$F(z) = \{y \in \mathbb{R}^N \mid z \in G(z + \sigma y)\}. \tag{5.3}$$

Then $G = (I + \sigma F)^{-1}$. Indeed, we get

$$\begin{aligned} x \in G(y) \quad &\Longleftrightarrow \quad \frac{y - x}{\sigma} \in F(x), \\ &\Longleftrightarrow \quad y \in x + \sigma F(x), \\ &\Longleftrightarrow \quad x \in (I + \sigma F)^{-1}(y). \end{aligned}$$

Next we show that the map F is monotone. Let (z_1, y_1) and (z_2, y_2) belong to $\operatorname{gph}(F)$; then $z_i \in G(z_i + \sigma y_i), i = 1, 2$. Thereby we get

$$z_i \rightsquigarrow \Theta \text{ on } [\pi(z_i + \sigma y_i)\ t_f] \text{ for } i = 1, 2,$$

and thanks to (5.1) we get

$$\langle z_2 - z_1 + \sigma(y_2 - y_1), z_2 - z_1\rangle \geq \xi(z_2 - z_1 + \sigma(y_2 - y_1), z_2 - z_1).$$

Putting $u = z_2 - z_1$ and $v = y_2 - y_1$, and using (5.2), it follows that

$$\sigma\langle u, v\rangle \geq \xi(u + \sigma v, u) - |u|^2 \geq 0.$$

Hence $\langle z_2 - z_1, y_2 - y_1\rangle \geq 0$, and so the set-valued mapping F is monotone.

As a result, the map G is the resolvent of a monotone set-valued mapping, and therefore $G(y)$ is single-valued for all $y \in \operatorname{dom}(G)$; see [2, 16]. It follows that subset $\Sigma \cap J(\pi(y)$ is a singleton for all y such that $\pi(y) \in \operatorname{dom}(J)$. Thus pair (f, Θ) is Σ-observable on $[t_0\ t_f]$ for all $t_0 \in \pi(\mathbb{R}^N)$. ◻

Remark 5.2. Whenever function π is surjective, then pair (f, Θ) is Σ-observable on $[t_0\ t_f]$ for all t_0 and the function $t_0 \to z_0$ is Lipschitz continuous globally with constant $1/\sigma$; see [16, Proposition 12.54] relative to strongly monotone maps.

Next we show that it is possible to weaken the hypothesis of Theorem 5.1 by replacing condition (5.1) by

$$\begin{matrix} z_1 \rightsquigarrow \Theta \text{ on } [\pi(y_1)\ t_f] \\ z_2 \rightsquigarrow \Theta \text{ on } [\pi(y_2)\ t_f] \end{matrix} \Longrightarrow \langle y_2 - y_1, z_2 - z_1\rangle \geq 0 \tag{5.4}$$

for z_1, z_2 and y_1, y_2 in $\mathbb{R}^N$. In this case the map $J \circ \pi$ is monotone and by using [20, Theorem 1] we get the following result.

THEOREM 5.3. *Assume that* $\pi^{-1}(\operatorname{dom}(J))$ *has nonempty interior and that* (5.4) *holds for all* z_1, z_2 *in subset* Σ. *Then pair* (f, Θ) *is* Σ*-observable on horizon* $[t_0\ t_f]$ *for almost every* t_0 *in* $[0, t_f]$.

Proof. In [20, Theorem 1], it is stated that when the domain of a monotone set-valued mapping has nonempty interior, the set of points where it is not single-valued has a Lebesgue measure zero. This is actually satisfied by the map $J \circ \pi$. ◻

5.2. Using convexity and/or contingent derivative. There are also single-valuedness results where convexity plays a central role; see [9, 15] and the references therein, and see [2], where the contingent derivative is also used as a tool. The later allows for providing a characterization of observability in terms of sequences.

THEOREM 5.4. *Assume that* $J_\Sigma(t_0)$ *is convex and that the following statement holds for some* $z_0 \in J_\Sigma(t_0)$*:*

$$\begin{array}{l} (h_n, s_n) \to (0^+, 0) \text{ and } y_n \to y \text{ with} \\ z_0 + h_n y_n \in \Sigma \ \forall n \text{ and } \forall\, n\text{:} \\ z_0 + h_n y_n \rightsquigarrow \Theta \text{ on } [t_0 + h_n s_n \ t_f], \end{array} \implies y = 0; \tag{5.5}$$

then pair (f, Θ) *is* Σ*-observable on horizon* $[t_0 \ t_f]$*.*

Proof. Note that (5.5) can also be expressed in the form

$$\begin{array}{l} h_n \to 0^+ \text{ and } (s_n, y_n) \to (0, y) \text{ with} \\ z_0 + h_n y_n \in J_\Sigma(t_0 + h_n s_n) \ \forall\, n\text{:} \end{array} \implies y = 0,$$

which is equivalent to

$$\begin{array}{l} h_n \to 0^+ \text{ and } (y_n, s_n) \to (y, 0) \text{ with} \\ (z_0, t_0) + h_n(y_n, s_n) \in \mathrm{gph}(J_\Sigma^{-1}) \ \forall n \end{array} \implies y = 0.$$

This means that

$$(y, 0) \in T_{\mathrm{gph}(J_\Sigma^{-1})}(z_0, t_0) \implies y = 0,$$

whence we get

$$(y, 0) \in \mathrm{gph}(DJ_\Sigma^{-1}(z_0, t_0)) \implies y = 0.$$

That is to say $\ker DJ_\Sigma^{-1}(z_0, t_0) = \{0\}$. Thanks to the second part of [2, Theorem 5.4.7], we get $(J_\Sigma^{-1})^{-1}(t_0) = \{z_0\}$. This therefore yields $J_\Sigma(t_0) = \{z_0\}$. ☐

As a consequence of Theorem 5.4, we next derive a sufficient tangential condition for Σ-observability. We begin by introducing the following sequence of subsets:

$$\begin{array}{l} \Theta^{(0)} \doteq \Theta \text{ and for each } k \in \mathbb{N}, \\ \Theta^{(k+1)} \doteq \{(t, z) \in \Theta^{(k)} \mid (1, f(t, z)) \in T_{\Theta^{(k)}}(t, z)\}. \end{array} \tag{5.6}$$

By construction, these stand for a decreasing sequence of subsets.

COROLLARY 5.5. *Assume that* $J_\Sigma(t_0)$ *is convex and that the following statement holds for some* $z_0 \in J_\Sigma(t_0)$ *and* $m \in \mathbb{N}$*:*

$$\begin{array}{l} y \in T_\Sigma(z_0) \text{ and} \\ (0, y) \in T_{\Theta^{(m)}}(t_0, z_0) \end{array} \implies y = 0. \tag{5.7}$$

Then pair (f, Θ) *is* Σ*-observable on* $[t_0 \ t_f]$*.*

Proof. First, we claim that (t_0, z_0) belongs to all subsets $\Theta^{(\cdot)}$. Indeed, as $z_0 \in J(t_0)$, there exists a solution $\bar{z}$ which satisfies,

$$\bar{z}(t_0) = z_0 \text{ and } (t, \bar{z}(t)) \in \Theta \ \forall\, t \in [t_0 \ t_f].$$

By repetitively using the tangential condition of [1, Lemma 1.1.4], and taking into account (6.4), we infer that

$$(t, \bar{z}(t)) \in \Theta^{(k)} \ \forall\, t \in [t_0 \ t_f] \text{ and } k \in \mathbb{N},$$

and hence $(t_0, z_0) \in \Theta^{(k)}$ for all k.

Now, suppose that pair (f, Θ) is not Σ-observable on horizon $[t_0\ t_f]$; then Condition (5.5) does not hold. As a result, there exist a sequence $(h_n, s_n, y_n)_n$ which converges to $(0, 0, \bar{y})$, with $\bar{y} \neq 0$, and a sequence of system solutions $(\bar{z}_n)_n$ such that

$$\bar{z}_n(t_0 + h_n s_n) = z_0 + h_n y_n \in \Sigma\ \forall\ n.$$

Thereby $\bar{y} \in T_\Sigma(z_0)$. Furthermore, these solutions satisfy the following viability condition:

$$(t, \bar{z}_n(t)) \in \Theta\ \forall\ t \in [t_0 + h_n s_n\ t_f] \text{ and } \forall\ n.$$

As in the beginning of the proof, a repetitive use of [1, Lemma 1.1.4] yields

$$(t, \bar{z}_n(t)) \in \Theta^{(k)}\ \forall\ t \in [t_0 + h_n s_n\ t_f] \text{ and } k, n \in \mathbb{N}.$$

In particular we get, by taking $t \doteq t_0 + h_n s_n, k = m$ in the last expression,

$$(t_0, z_0) + h_n(s_n, y_n) \in \Theta^{(m)}\ \forall\ n.$$

This implies that $(0, \bar{y}) \in T_{\Theta^{(m)}}(t_0, z_0)$. With $\bar{y} \in T_\Sigma(z_0)$ and $\bar{y} \neq 0$, we get a contradiction of implication (5.7). ☐

Remark 5.6. Toward applications of Corollary 5.5, we notice the following important facts:

(i) An instance for which condition (6.5) holds can be provided by almost countability of subset $\Theta_\Sigma^{(m)}$. Indeed, this implies that its contingent cone is reduced to the origin at each of its elements. As a result, for all $(t_0, z_0) \in \Theta^{(m)} \cap \operatorname{gph}(J_\Sigma)$, pair (f, Θ) is Σ-observable on $[t_0\ t_f]$.

(ii) Subset Θ alone may provide an Σ-observable pair with any field f. In fact, (5.7) could hold for $m = 0$.

Example 5.7. Consider the continuous time-varying linear system

$$\dot{z} = A(t)z + b(t) \qquad \text{for } 0 \leq t \leq t_f, \tag{5.8}$$

where $A(\cdot)$ and $b(\cdot)$ are functions of class $\mathcal{C}^N(0, t_f)$ with values respectively in $\mathcal{L}(\mathbb{R}^N)$ and $\mathbb{R}^N$. Assume that the output domain Θ is convex.

Let $t_0 \in [0\ t_f)$; then $J(t_0)$ is convex. In fact, let z_1 and z_2 be two elements of $J(t_0)$ and denote by $\bar{z}_1$ and $\bar{z}_2$ the corresponding solutions which satisfy

$$(t, \bar{z}_i(t)) \in \Theta \text{ for } i = 1, 2.$$

Let α, β belong to $[0\ 1]$ and satisfy $\alpha + \beta = 1$. As subset Θ is convex, we can easily see that $\alpha\bar{z}_1 + \beta\bar{z}_2$ is a solution of system (5.8) which yields $\alpha z_1 + \beta z_2 \in J(t_0)$.

Let Θ be equal to the hyperplane Φ of $\mathbb{R} \times \mathbb{R}^N$, given through reals δ, γ and a nonnull vector $u \in \mathbb{R}^N$ by

$$\Phi \doteq \{(t, z) \in \mathbb{R} \times \mathbb{R}^N \mid \delta t + \langle u, z\rangle = \gamma\}. \tag{5.9}$$

Note Φ has an empty interior, as well as all its subsets $\Phi^{(k)}$ given by (5.6). These can be estimated by first observing that

$$T_\Phi(t, z) = \{(s, y) \in \mathbb{R} \times \mathbb{R}^N \mid \delta s + \langle u, y\rangle = 0\}\ \forall\ (t, z) \in \Phi.$$

It follows that

$$\Phi^{(1)} = \{(t, z) \mid \delta t + \langle u, z\rangle = \gamma \text{ and } \delta + \langle u, A(t)z + b(t)\rangle = 0\}.$$

For $m \geq 2$, we claim that

$$\begin{aligned} \Phi^{(m)} \subset \{(t,z) \mid \delta t + \langle u, z\rangle = \gamma,\ \delta + \langle u, A(t)z + b(t)\rangle = 0,\\ \textit{and } \langle u, A^{(k)}(t)z + b^{(k)}(t)\rangle = 0 \text{ for } k = 2, \dots, m\}, \end{aligned} \tag{5.10}$$

where functions $A^{(k)}$ and $b^{(k)}$ are given for $k \geq 1$ through the scheme

$$\begin{aligned} A^{(k+1)} &= \dot{A}^{(k)} + A^{(k)}A \quad \text{with } A^1 = A,\\ b^{(k+1)} &= \dot{b}^{(k)} + A^{(k)}b \quad \text{with } b^1 = b. \end{aligned} \tag{5.11}$$

Indeed, for $m = 1$, expression (5.10) is true. Suppose that it holds for m, and let $\mu^{(k)}$ denote the function defined by $\mu^{(k)}(t,z) \doteq \langle u, A^{(k)}(t)z + b^{(k)}(t)\rangle$ for all t, z; then we get

$$d\mu^{(k)}(t,z)(s,y) = s\langle u, \dot{A}^{(k)}(t)z + \dot{b}^{(k)}(t)\rangle + \langle u, A^{(k)}(t)y)\rangle \ \forall\ (s,y).$$

Then, picking $s \doteq 1$ and $y = A(t)z + b(t)$, we use (5.11) to get

$$\begin{aligned} d\mu^{(k)}(t,z)(1, A(t)z + b(t)) &= \langle u, \dot{A}^{(k)}(t)z + \dot{b}^{(k)}(t)\rangle + \langle u, A^{(k)}(t)(A(t)z + b(t))\rangle\\ &= \langle u, A^{(k+1)}(t)z + b^{(k+1)}(t)\rangle. \end{aligned}$$

As a result, for all $(t,z) \in \Phi^{(m+1)}$, we get

$$(1, A(t)z + b(t)) \in T_{\Phi^{(m+1)}}(t,z) \implies \begin{array}{l} \delta + \langle u, A(t)z + b(t)\rangle = 0 \qquad \text{and}\\ \langle u, A^{(k)}(t)z + b^{(k)}(t)\rangle = 0 \text{ for } k = 2, \dots, m+1. \end{array}$$

Consequently, (5.10) holds for all $m \geq 2$.

Next, we proceed to apply Corollary 5.5. Let $t_0 \in [0\ t_f)$ such that $J(t_0) \neq \emptyset$ and $z_0 \in J(t_0)$. Out of expression (5.10), we get

$$(0,y) \in T_{\Phi^{(m)}}(t_0, z_0) \implies \begin{array}{l} \langle u, y\rangle = \langle u, A^{(k)}(t_0)y\rangle = 0\\ \text{for } k = 1, \dots, m. \end{array}$$

Whence, (5.7) can be set as follows:

$$(0,y) \in T_{\Phi^{(m)}}(t_0, z_0) \implies \begin{array}{l} \langle u, y\rangle = \langle {A^{(k)}}'(t_0)u, y\rangle = 0\\ \text{for } k = 1, \dots, m. \end{array}$$

As a conclusion, it turns out that the pair given by linear system (5.8) and the output domain Φ given by (5.9) is observable on $[t_0\ t_f]$ providing that the following condition holds:

$$\operatorname{rank}\left(u, A'(t_0)u, \dots, {A^{(N-1)}}'(t_0)u\right) = N.$$

Because it fits our context, we also consider a result [15, Theorem 3] which states conditions under which midconvex set-valued mappings have singleton images. Let Σ be a nonempty subset of $\mathbb{R}^N$, and define the set-valued mapping

$$J_\Sigma \doteq J(\cdot) \cap \Sigma,$$

where J is as in (4.1). Next, we consider the following implication:

$$\begin{array}{l} z_1 \rightsquigarrow \Theta \text{ on } [t_1\ t_f)\\ z_2 \rightsquigarrow \Theta \text{ on } [t_2\ t_f) \end{array} \implies \frac{z_1 + z_2}{2} \rightsquigarrow \Theta \text{ on } \left[\frac{t_1 + t_2}{2}\ t_f\right) \tag{5.12}$$

for all z_1, z_2 in Σ and t_1, t_2 in $[0, t_f)$ with t_f possibly equal to infinity. Then we are ready to show the following.

Theorem 5.8. *Let* Σ *be convex, and assume that* (5.12) *is satisfied. Suppose that there exist an open interval* $\mathcal{J} \subset \operatorname{dom}(J_\Sigma)$ *and an instant* $\bar{t} \in \mathcal{J}$ *such that pair* (f, Θ) *is* Σ*-observable on horizon* $[\bar{t}\ t_f)$. *Then the following statements hold:*

(i) *Pair* (f, Θ) *is* Σ*-observable on* $[t_0\ t_f)$ *for all* $t_0 \in \mathcal{J}$.

(ii) *There exist* $y_0 \in \mathbb{R}^N$ *and an additive function* $\xi : \mathcal{J} \to \Sigma - y_0$ *such that*

$$\xi(t_0) + y_0 \rightsquigarrow \Theta \textit{ on } [t_0\ t_f] \textit{ for each } t_0 \in \mathcal{J}.$$

Proof. Consider set-valued mapping $J_\Sigma|_{\mathcal{J}}$; then we observe that condition (5.12) expresses that it is midconvex, i.e.,

$$\frac{J_\Sigma(t_1) + J_\Sigma(t_2)}{2} \subset J_\Sigma\left(\frac{t_1 + t_2}{2}\right) \forall\, t_1, t_2 \in \mathcal{J}.$$

Since it has a singleton image (at instant $\bar{t}$), statements (i) and (ii) are immediate from using [15, Theorem 3]. □

Example 5.9. Let $f(t, z) \doteq Az$ with $A \in \mathcal{L}(\mathbb{R}^N)$, and assume that subsets Σ and Θ are convex. Then the implication given by (5.12) is satisfied, with $t_f \doteq \infty$. To show this, let t_1, t_2, z_1, z_2 as in the left-hand side of that implication, and denote by $\bar{z}_1, \bar{z}_2$ the corresponding solutions. Therefore,

$$(t, \bar{z}_i(t)) \in \Theta\ \forall\, t \in [t_i\ \infty) \text{ and } i = 1, 2.$$

Define the following function by

$$\bar{z}_3(t) \doteq \frac{1}{2}\left(\bar{z}_1\left(t - \frac{t_2 - t_1}{2}\right) + \bar{z}_2\left(t + \frac{t_2 - t_1}{2}\right)\right) \quad \forall\, t \in \left[\frac{t_1 + t_2}{2}\ \infty\right),$$

It stands for a solution of system $\dot{z} = Az$ and satisfies

$$\bar{z}_3\left(\frac{t_1 + t_2}{2}\right) = \frac{z_1 + z_2}{2},$$

and by expressing t as

$$t = \left(t - \frac{t_2 - t_1}{2}\right) + \left(t + \frac{t_2 - t_1}{2}\right) \quad \forall\, t \in \left[\frac{t_1 + t_2}{2}\ \infty\right),$$

and using the convexity of subset Θ, it follows that

$$(t, \bar{z}_3(t)) \in \Theta\ \forall\, t \in \left[\frac{t_1 + t_2}{2}\ \infty\right).$$

As a consequence of Theorem 5.8, for convex subsets Θ and Σ, whenever a pair (A, Θ) is Σ-observable on horizon $[\bar{t}\ \infty)$ for some $\bar{t} > 0$, then it is so on all horizons $[t_0\ \infty)$ for $t_0 > 0$.

6. Local observability.

To create a trivial instance of a locally observable pair around a point z_0 on horizon $[t_0\ t_f]$, let subset $J(t_0)$ be closed and z_0 belong to the open subset $J(t_0)^c$. Hence, the later contains a ball B centered at point z_0, and so $B \cap J(t_0)$ coincides with emptyset.

Also, we notice that Theorems 5.1, 5.3, and 5.8 and, more generally, any Σ-observability result can be used to establish conditions under which a pair is locally observable around a state z_0. Actually it suffices to check whether the statements of these results hold when subset Σ can be taken as a ball centered at point z_0.

6.1. A negative result. Due to Peano's theorem (see section 2 for more details), it can be noticed that in most cases, whenever a point (t_0, z_0) belongs to the interior of the output domain, the pair under consideration is not locally observable around z_0 on $[t_0\ t_f]$.

For a point (t_0, z_0) belonging to the boundary $\partial\Theta$, an immediate application of a result stated in [1, section 4.3.3] shows how a pair can first be checked to be nonlocally observable near that point.

THEOREM 6.1. *Let* $(t_0, z_0) \in \partial\Theta$, *and assume that there exists* $r > 0$ *such that*

$$(1, f(t, z)) \in T_\Theta(t, z)\ \forall\ (t, z) \in \partial\Theta \cap B_r(t_0, z_0). \tag{6.1}$$

Then pair (f, Θ) *is not locally observable around* z_0 *on some horizon* $[t_0\ t_f]$.

Proof. According to the above mentioned result, (6.1) implies that system

$$(\dot{t}, \dot{z}) = (1, f(t, z))$$

has a viable solution in subset Θ, which is issued from (t_0, z_0) and defined on some interval $[t_0\ t_1]$. As a result, $z_0 \rightsquigarrow \Theta$ on $[t_0, t_1]$.

Now, let ρ and z_1 be such that

$$0 < |z_1 - z_0| < \rho < \frac{r}{2} \text{ and } (t_0, z_1) \in \partial\Theta;$$

then, by virtue of the triangle inequality, (6.1) holds true with $(z_1, r/2)$ instead of (z_0, r). Consequently, there exists $t_2 > t_0$ such that $z_1 \rightsquigarrow \Theta$ on $[t_0, t_2]$, and so both z_0 and z_1 are distinct indistinguishable elements of the ball $B(z_0, \rho)$ on horizon $[t_0\ \inf(t_1, t_2)]$. ▯

Example 6.2. Consider the unforced pendulum equation,

$$\ddot{\alpha} - a\dot{\alpha} - b\sin(\alpha) = 0 \qquad (a \geq 0, b > 0) \tag{6.2a}$$

with output given by

$$t^2 + \alpha^2 + \dot{\alpha}^2 \leq 2. \tag{6.2b}$$

Here, the output domain is $\Theta \doteq \bar{B}(0, 2)$ and function f is defined by

$$f(t, \alpha, \dot{\alpha}) = (\dot{\alpha}, a\dot{\alpha} + b\sin(\alpha)).$$

To check whether (6.1) holds, we first note that

$$(1, \dot{\alpha}, a\dot{\alpha} + b\sin(\alpha)) \in T_\Theta(t, \alpha, \dot{\alpha}) \quad \text{iff} \quad t + \alpha\dot{\alpha} + \dot{\alpha}(a\dot{\alpha} + b\sin(\alpha)) \leq 0$$

for all $(t, \alpha, \dot{\alpha})$ in $\partial\Theta$. Now, let $(t_0 \doteq 0, \alpha_0 \doteq 1, \dot{\alpha}_0 \doteq -1)$ be an element of $\partial\Theta$; then (6.1) can be stated, for some $r > 0$, as follows:

$$t + \dot{\alpha}(\alpha + a\dot{\alpha} + b\sin(\alpha)) \leq 0\ \forall\ (t, \theta, \dot{\theta}) \in \partial\Theta \cap B_r(0, 1, -1).$$

Actually, such real r does exist for $a - b\sin(1) < 1$, because of continuity of the last expression, which takes the value $a - b\sin(1) - 1$ for $(t, \alpha, \dot{\alpha}) = (0, 1, -1)$.

As a conclusion, for $a - b\sin(1) < 1$, the pair given by (6.2) is not locally observable near point $(1, -1)$ on some horizon $[0\ t_f]$ with $t_f > 1$.

6.2. By means of contingent derivative. Next, by using a property that holds the contingent derivative in checking local uniqueness, we prove a new result, characterizing local observability in terms of sequences.

THEOREM 6.3. *Let* $(t_0, z_0) \in \mathrm{gph}(J)$ *and assume that the following statement holds:*

$$\begin{array}{l} (h_n, s_n) \to (0^+, 0) \text{ and } y_n \to y \text{ with} \\ z_0 + h_n y_n \rightsquigarrow \Theta \text{ on } [t_0 + h_n s_n \ t_f] \ \forall n \end{array} \implies y = 0; \tag{6.3}$$

then pair (f, Θ) *is locally observable around* z_0*, on horizon* $[t_0 \ t_f]$*.*

Proof. Indeed, (6.3) can also be expressed as follows:

$$\begin{array}{l} h_n \to 0^+ \text{ and } (y_n, s_n) \to (y, 0) \text{ with} \\ (z_0, t_0) + h_n(y_n, s_n) \in \mathrm{gph}(J^{-1}) \ \forall n \end{array} \implies y = 0.$$

This means that

$$(y, 0) \in T_{\mathrm{gph}(J^{-1})}(z_0, t_0) \implies y = 0,$$

whence we get

$$(y, 0) \in \mathrm{gph}(DJ^{-1}(z_0, t_0)) \implies y = 0.$$

That is to say $\ker DJ^{-1}(z_0, t_0) = \{0\}$. By virtue of [2, Theorem 5.4.7], there exists $\epsilon > 0$ such that $(J^{-1})^{-1}(t_0) \cap B(z_0, \epsilon) = \{z_0\}$. This therefore yields

$$J(t_0) \cap B(z_0, \epsilon) = \{z_0\}.$$

That ends the proof. ☐

As a consequence of Theorem 6.3, we next establish a sufficient tangential condition for local observability. It involves the sequence of subsets which are introduced by (5.6) and can be expressed here, as follows:

$$\begin{array}{l} \Theta^{(0)} \doteq \Theta \text{ and for each } k \in \mathbb{N} \\ \Theta^{(k+1)} \doteq \{(t, z) \in \Theta^{(k)} \mid (1, f(t, z)) \in T_{\Theta^{(k)}}(t, z)\}. \end{array} \tag{6.4}$$

COROLLARY 6.4. *Let* $(t_0, z_0) \in \mathrm{gph}(J)$*, and assume that for some* $m \in \mathbb{N}$*, the following statement holds:*

$$(0, y) \in T_{\Theta^{(m)}}(t_0, z_0) \implies y = 0. \tag{6.5}$$

Then pair (f, Θ) *is locally observable around* z_0 *on horizon* $[t_0 \ t_f]$*.*

Proof. First, we claim that (t_0, z_0) belongs to all subsets $\Theta^{(\cdot)}$. Indeed, as $z_0 \in J(t_0)$, there exists a solution $\bar{z}$ which satisfies

$$\bar{z}(t_0) = z_0 \text{ and } (t, \bar{z}(t)) \in \Theta \ \forall \ t \in [t_0 \ t_f].$$

By repetitively using the tangential condition of [1, Lemma 1.1.4], and taking into account (6.4), we infer that

$$(t, \bar{z}(t)) \in \Theta^{(k)} \ \forall \ t \in [t_0 \ t_f] \text{ and } k \in \mathbb{N},$$

and hence $(t_0, z_0) \in \Theta^{(k)}$ for all k.

Now, suppose that pair (f, Θ) is not locally observable around z_0 on horizon $[t_0\ t_f]$; then Condition (6.3) does not hold. As a result, there exist a sequence $(h_n, s_n, y_n)_n$ which converges to $(0, 0, \bar{y})$, with $\bar{y} \neq 0$, and a sequence of system solutions $(\bar{z}_n)_n$ such that, $\bar{z}_n(t_0 + h_n s_n) = z_0 + h_n y_n$ for all n, and the following viability condition holds for all n:

$$(t, \bar{z}_n(t)) \in \Theta\ \forall\ t \in [t_0 + h_n s_n\ t_f]\,.$$

As in the beginning of the proof, a repetitive use of [1, Lemma 1.1.4] yields

$$(t, \bar{z}_n(t)) \in \Theta^{(k)}\ \forall\ t \in [t_0 + h_n s_n\ t_f] \text{ and } k, n \in \mathbb{N}.$$

In particular we get, by taking $t \doteq t_0 + h_n s_n, k = m$ in the last expression,

$$(t_0, z_0) + h_n(s_n, y_n) \in \Theta^{(m)}\ \forall\ n.$$

This implies that $(0, \bar{y}) \in T_{\Theta^{(m)}}(t_0, z_0)$, and $\bar{y} \neq 0$ yields a contradiction. ∎

By the result [2, Theorem 5.4.7] itself, we can infer that whenever the map J has the Aubin property [16] at point (t_0, z_0), then condition (6.3) is also necessary in order pair (f, Θ) be locally observable around z_0 on horizon $[t_0\ t_f]$. Here, this Aubin property holds that there exists a constant $\rho > 0$ such that the following statement holds:

$$\begin{aligned} &\forall\ \epsilon \text{ there exist } \eta > 0 \text{ such that} \\ &\forall\ t, s \in \left(t_0 - \frac{\eta}{2}\ t_0 + \frac{\eta}{2}\right) \text{ and } z \in J(t) \cap B(z_0, \eta), \\ &\text{one has } d(z, J(s)) \leq (\rho + \epsilon)|s - t|. \end{aligned} \tag{6.6}$$

This leads us to derive a necessary condition, which may be more easily checked than the condition of (6.6).

PROPOSITION 6.5. *Let* $(t_0, z_0) \in \operatorname{gph}(J)$, *and assume that the regularity assumption* (6.6) *is satisfied. In order for pair* (f, Θ) *be locally observable around* z_0, *on horizon* $[t_0\ t_f]$, *it is necessary that the following statement holds:*

$$\left.\begin{array}{l} K \subset \Theta, \text{ viable under } g, \\ (t_0, z_0) \in K, \text{ and} \\ (0, y) \in T_K(t_0, z_0) \end{array}\right. \implies y = 0, \tag{6.7}$$

where $g = (1, f)'$.

Proof. Suppose that assertion (6.7) is false; then there exist a viable subset under mapping g, $K \subset \Theta$, and y_0 such that

$$y_0 \neq 0 \text{ and } (0, y_0) \in T_K(t_0, z_0).$$

As a result, there is a sequence $(h_n, s_n, y_n)_n$ such that $(h_n, s_n, y_n) \to (0, 0, y_0)$ and

$$(t_0 + h_n s_n, z_0 + h_n y_n) \in K\ \forall\ n.$$

As subset K is viable under g, for each n, there exists a system solution $\bar{z}_n$ which satisfies the following statement:

$$\bar{z}_n(t_0 + h_n s_n) = z_0 + h_n y_n \text{ and } (t, \bar{z}_n(t)) \in K\ \forall\ t \in [t_0 + h_n s_n\ t_f].$$

Because subset K is included in subset Θ, condition (6.3) does not hold, and, thanks to assumption (6.6), pair (f, Θ) cannot be locally observable around z_0 on horizon $[t_0\ t_f]$. ∎

6.3. Local observability for output given by equalities and inequalities. Assume that the output domain is given through expressions of the form

$$\begin{aligned}\Theta_\mu \doteq \{(t,z) \in \mathbb{R}^+ \times \mathbb{R}^N \mid \mu_i(t,z) = 0, i = 1,\dots,p,\\ \text{and } \mu_j(t,z) \geq 0 \text{ for } j = p+1,\dots,q\},\end{aligned} \tag{6.8}$$

where p and q are integers, with $q \geq p$, and $\mu = (\mu_1,\dots,\mu_q)'$ denotes a continuous $\mathbb{R}^q$-valued mapping. One instance that is described by this setting consists of the standard output equation (1.2b), in which case $p = q = n$ and $\mu \doteq h - \theta$. In order to get estimates of subsets $\Theta_\mu^{(m)}$ provided by (6.4), we need to assume that output domain Θ_μ has an empty interior, i.e.,

$$\begin{aligned}&\forall\ (t,z) \in \Theta_\mu \text{ and } r > 0, \text{ there is } (s,y) \in B_r(t,z)\\ &\text{such that } \mu_i(s,y) \neq 0 \text{ for some } i \text{ or } \mu_j(s,y) < 0 \text{ for some } j.\end{aligned} \tag{6.9}$$

Also, assume that function μ is smooth enough to allow the well-definiteness of the Lie derivatives,

$$\begin{aligned}\mu_i^{(0)} &\doteq \mu_i,\\ \mu_i^{(k+1)} &\doteq \frac{\partial \mu_i^{(k)}}{\partial t} + \frac{\partial \mu_i^{(k)}}{\partial z} f,\end{aligned} \tag{6.10a}$$

for $k \in \mathbb{N}$ and $i = 1,\dots,p$. Then we build the sequence of subsets by

$$\begin{aligned}\mathcal{X}_m \doteq \{(t,z) \mid \mu_i^{(k)} = 0 \text{ for } i = 1,\dots,p \textit{ and } k = 0,\dots,m,\\ \text{and } \mu_j \geq 0 \text{ for } j = p+1,\dots,q\}\end{aligned} \tag{6.10b}$$

for all $m \in \mathbb{N}$, and we introduce the set-valued mapping, defined for all (t,z) in subset Θ by

$$K_m(t,z) \doteq \bigcap_{\substack{i=1,p\\ k=0,m}} \ker \frac{\partial \mu_i^{(k)}}{\partial z}(t,z)\ \cap \bigcap_{j \in I(t,z)} \frac{\partial \mu_j}{\partial z}(t,z)^+, \tag{6.10c}$$

where,

$$I(t,z) \doteq \{j = p+1,\dots,q \mid \mu_j(t,z) = 0\}.$$

Then, applying Corollary 6.4 yields the result below.

THEOREM 6.6. *The following statements hold:*

(i) *Let* $(t_0,z_0) \in \mathrm{gph}(J)$ *be such that* $K_m(t_0,z_0) = \{0\}$ *for some* m*; then pair* (f,Θ_μ) *is locally observable near* z_0 *on horizon* $[t_0\ t_f]$*.*

(ii) *Whenever for some* $m \in \mathbb{N}$*, subset* $\mathcal{X}_m$ *is almost countable, then for all* $(t_0,z_0) \in \mathrm{gph}(J)$*, pair* (f,Θ_μ) *is locally observable near* z_0 *on horizon* $[t_0\ t_f]$*.*

Proof. We begin by showing that

$$\Theta_\mu^{(m)} \subset \mathcal{X}_m\ \forall\ m \in \mathbb{N}. \tag{6.11}$$

Indeed, inclusion (6.11) is true for $m = 0$ because it stands for an equality. Suppose the inclusion be true for m. It follows that

$$\begin{aligned}\Theta_\mu^{(m+1)} &= \{(t,z) \in \Theta_\mu^{(m)} \mid (1, f(t,z)) \in T_{\Theta_\mu^{(m)}}(t,z)\}\\ &\subset \{(t,z) \in \mathcal{X}_m \mid (1, f(t,z)) \in T_{\mathcal{X}_m}(t,z)\}.\end{aligned}$$

Since subset $\mathcal{X}_m$ is included in Θ, it has an empty interior too. Therefore,

$$\begin{aligned}\Theta_\mu^{(m+1)} &\subset \{(t,z) \in \partial\mathcal{X}_m \mid (1, f(t,z)) \in T_{\mathcal{X}_m}(t,z)\}\\ &\subset \left\{(t,z) \mid \frac{\partial\mu_i^{(k)}}{\partial t} + \frac{\partial\mu_i^{(k)}}{\partial z} f = 0 \text{ for } i = 1,\dots,p \textit{ and } k = 0,\dots,m,\right.\\ &\qquad\qquad \left.\text{and } \mu_j \geq 0 \text{ for } j = p+1,\dots,q\right\}.\end{aligned}$$

Thanks to (6.10a) and (6.10b), we get $\Theta_\mu^{(m+1)} \subset \mathcal{X}_{m+1}$.

Now, let (t_0, z_0) as in (i); then it belongs to subset $\Theta_\mu^{(m)}$ and by inclusion (6.11) it is an element of $\mathcal{X}_m$. As a result,

$$T_{\Theta_\mu^{(m)}}(t_0,z_0) \subset T_{\mathcal{X}_m}(t_0,z_0).$$

To check implication (6.5), we then get

$$(0,y) \in T_{\Theta_\mu^{(m)}}(t_0,z_0) \implies \begin{array}{l}\dfrac{\partial\mu_i^{(k)}}{\partial z}(t_0,z_0)y = 0 \text{ for } i = 1,\dots,p, \text{ and } k = 1,\dots,m,\\[2ex] \dfrac{\partial\mu_j}{\partial z}(t_0,z_0)y \geq 0 \ \forall\, j \in I(t_0,z_0),\end{array}$$

whence (6.10c) yields

$$(0,y) \in T_{\Theta_\mu^{(m)}}(t_0,z_0) \implies y \in K(t_0,z_0).$$

Then we can conclude assertion (i) by virtue of Corollary 6.4. As regards (ii), we see that subset $\Theta_\mu^{(m)}$ is almost countable too, and it follows that its contingent cone coincides to $\{0\}$ at each of its elements. □

Example 6.7. Let us get back to the pendulum equation (6.2a) and assume that the output domain is described by

$$\alpha + \dot\alpha = 3t,\ \alpha\dot\alpha \geq 2 \ \text{ and } \ \alpha^2 + \dot\alpha^2 \geq 5t^2. \tag{6.12}$$

Let

$$\mu_1 = \alpha + \dot\alpha - 3t,\ \mu_2 = \alpha\dot\alpha - 2,\ \mu_3 = \alpha^2 + \dot\alpha^2 - 5t^2, \quad \text{and } (t_0, \alpha_0, \dot\alpha_0) = (1,1,2).$$

For that point, one easily gets $\mu_1 = \mu_2 = \mu_3 = 0$, and so $I(t_0,\alpha_0,\dot\alpha_0) = \{2,3\}$, and the partial derivatives of functions μ_i with respect to $z = (\alpha, \dot\alpha)$ are given for $y = (y_1, y_2)$ by

$$\frac{\partial\mu_1}{\partial z}y = y_1 + y_2, \quad \frac{\partial\mu_2}{\partial z}y = 2y_1 + y_2, \quad \text{and } \frac{\partial\mu_3}{\partial z}y = 2(y_1 + 2y_2).$$

It follows that $K_0(t_0,\alpha_0,\dot\alpha_0) = \{0\}$. Moreover, by executing an ODE solver, it turns out that initial data $(1,2)$ generates the output given by (6.12) on horizon $[1\ 1.4]$. Thanks to Theorem 6.6 we finally conclude that the pair constituted of system (6.2a) and output (6.12) is locally observable near point $(1,2)$ on horizon $[1\ 1.4]$.

A relevant issue to address is how the standard output equation (1.2b) can be examined in light of the previous results. Actually, we recover the rank test for nonlinear local observability.

COROLLARY 6.8. *Let* Θ_s *be given as in* (1.2b)*; then pair* (f, Θ_s) *is locally observable near a point* z_0 *on* $[t_0\ t_f]$*, provided that* $z_0 \in J(t_0)$*, and*

$$\operatorname{rank}\left[\frac{\partial h}{\partial z}(t_0, z_0) \quad \frac{\partial h^{(1)}}{\partial z}(t_0, z_0) \ \ldots \ \frac{\partial h^{(m)}}{\partial z}(t_0, z_0)\right] = N \tag{6.13}$$

for some $m \in \mathbb{N}$.

Proof. By letting, $\mu \doteq h - \theta$ in (6.10c), we get

$$K_m(t_0, z_0) = \bigcap_{\substack{i=1,p \\ k=0,m}} \ker \frac{\partial h_i^{(k)}}{\partial z}(t_0, z_0) = \bigcap_{k=0,m} \ker \frac{\partial h^{(k)}}{\partial z}(t_0, z_0)$$

for $m \in \mathbb{N}$. Noting that rank condition (6.13) holds true iff $K_m(t_0, z_0) = \{0\}$, we invocate assertion (i) of Theorem (6.6) to end the proof. ☐

6.4. Local observability by local injectivity. Below, we use a set-valued concept of local injectivity in order to get a supplemental result on local observability. We first need to to prove the following lemma.

LEMMA 6.9. *There exists* $W \in \mathcal{N}(z_0)$ *such that pair* (f, Θ) *is* W*-observable on* $[t_0\ t_f]$ *for all* t_0 *iff the set-valued mapping* J^{-1} *is locally injective around point* z_0.

Proof. We have the following statements:

$$\begin{aligned} J^{-1} \text{ is not loc. injective around } z_0 &\Longleftrightarrow \forall\, W \in \mathcal{N}(z_0),\ \exists t_0 \in [0\ t_f)\,, \exists (z_1, z_2) \in W^2 \\ &\qquad \text{s.t. } z_1 \neq z_2 \text{ and } t_0 \in J^{-1}(z_1) \cap J^{-1}(z_2), \\ &\Longleftrightarrow \forall\, W \in \mathcal{N}(z_0),\ \exists t_0 \in [0\ t_f)\,, \exists (z_1, z_2) \in W^2 \\ &\qquad \text{s.t. } z_1 \neq z_2, \text{ and } z_1, z_2 \rightsquigarrow \Theta \text{ on } [t_0\ t_f)\,, \\ &\Longleftrightarrow \forall\, W \in \mathcal{N}(z_0),\ \exists t_0 \in [0\ t_f) \text{ s.t.} \\ &\qquad (f, \Theta) \text{ is not } W\text{-observable on } [t_0\ t_f]\,. \end{aligned}$$

☐

Next we apply [2, Theorem 5.4.10] to establish conditions under which the map J^{-1} is locally injective. For a set-valued mapping F from a finite dimensional space X to a normed space Y, this theorem stipulates that F is locally injective around an element $x_0 \in \operatorname{dom}(F)$ whenever the following statements hold:

(i) $\operatorname{gph}(F)$ is closed,
(ii) $F(B(x_0, \eta))$ is relatively compact for some $\eta > 0$,
(iii) $\ker PF(x_0, y) = \{0\}$ for all $y \in F(x_0)$,

where $PF(\cdot, \cdot)$ is the paratingent derivative as introduced by (1.3).

THEOREM 6.10. *Let* $\operatorname{gph}(J)$ *be closed and let* z_0 *belong to* $\operatorname{dom}(J^{-1})$*. Furthermore, assume that the following statement holds:*

$$\begin{array}{l} z_0 \in J(t_0), z_n \to z_0 \ \textit{and} \\ (h_n, s_n) \to (0^+, 0),\ y_n \to y \ \textit{with} \\ z_n + h_n y_n \rightsquigarrow \Theta \ \textit{on} \ [t_0 + h_n s_n\ t_f] \ \forall n \end{array} \quad \Longrightarrow \ y = 0. \tag{6.14}$$

Then pair (f, Θ) *is locally observable near* z_0 *on all horizons* $[t_0\ t_f]$ *such that* z_0 *belongs to subset* $J(t_0)$.

Proof. Thanks to Lemma 6.9, it suffices to show that J^{-1} is locally injective around z_0. To that end, we proceed by checking statements (i), (ii), and (iii) relatively to $F \doteq J^{-1}$. Obviously (i) and (ii) hold. Concerning (iii), we note that (6.14) can be rewritten as

$$\left.\begin{array}{l} z_0 \in J(t_0), z_n \to z_0 \text{ and} \\ h_n \to 0^+, \text{ and } (y_n, s_n) \to (y, 0) \text{ with} \\ (z_n, t_0) + h_n(y_n, s_n) \in \mathrm{gph}(J^{-1})\ \forall n \end{array}\right. \implies y = 0,$$

and by definition of the paratingent cone, we get

$$\left.\begin{array}{l} z_0 \in J(t_0) \\ (y, 0) \in P_{\mathrm{gph}(J^{-1})}(z_0, t_0) \end{array}\right. \implies y = 0.$$

On another hand, we have

$$\begin{array}{rcl} (y, 0) \in P_{\mathrm{gph}(J^{-1})}(z_0, t_0) & \iff & (y, 0) \in \mathrm{gph}(PJ^{-1}(z_0, t_0)) \\ & \iff & y \in \ker PJ^{-1}(z_0, t_0). \end{array}$$

It follows that $\ker PJ^{-1}(z_0, t_0) = \{0\}$ for all $t_0 \in J^{-1}(z_0)$. ☐

It is of interest to notice that due to the proof of Lemma 6.9, the neighborhood of z_0 that derives local observability of pair (f, Θ) can be taken the same for all the horizons which are mentioned in Theorem 6.10.

Below, we use subsets $\Theta^{(\cdot)}$ of (6.4) in order to prove a sufficient condition involving the paratingent cone.

COROLLARY 6.11. *Let $z_0 \in \mathrm{dom}(J^{-1})$, and assume that for some $m \in \mathbb{N}$, the following statement holds:*

$$\left.\begin{array}{l} z_0 \in J(t_0) \textit{ and} \\ (0, y) \in P_{\Theta^{(m)}}(t_0, z_0) \end{array}\right. \implies y = 0. \tag{6.15}$$

Then pair (f, Θ) is locally observable around z_0 on all horizons $[t_0\ t_f]$ for which $z_0 \in J(t_0)$.

Proof. As in the beginning of the proof of Corollary 6.5, we observe that if $t_0 \in J^{-1}(z_0)$, then couple (t_0, z_0), belongs to all subsets $\Theta^{(\cdot)}$.

Now, suppose that pair (f, Θ) is not locally observable around z_0 on horizon $[t_0\ t_f]$ for some $t_0 \in J^{-1}(z_0)$. As a result, Condition (6.14) does not hold. Thereby, there exist a sequence $(h_n, s_n, y_n, z_n)_n$ which converges to $(0, 0, \bar{y}, z_0)$, with $\bar{y} \neq 0$, and a sequence of system solutions $(\bar{z}_n)_n$ such that $\bar{z}_n(t_0 + h_n s_n) = z_n + h_n y_n$, for all n, and the following viability condition holds for all n:

$$(t, \bar{z}_n(t)) \in \Theta\ \forall\ t \in [t_0 + h_n s_n\ t_f]\,.$$

Hence, we get

$$(t, \bar{z}_n(t)) \in \Theta^{(k)}\ \forall\ t \in [t_0 + h_n s_n\ t_f] \text{ and } k, n \in \mathbb{N}.$$

In particular, we pick $t \doteq t_0 + h_n s_n, k = m$ in the last expression to obtain

$$(t_0, z_n) + h_n(s_n, y_n) \in \Theta^{(m)}\ \forall\ n.$$

This implies that $(0, \bar{y}) \in P_{\Theta^{(m)}}(t_0, z_0)$, and $\bar{y} \neq 0$ yields a contradiction. ☐

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