Graphical Analysis of the Riemann Zeta Function

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Abstract

The purpose of the experiment was to investigate and explain the graphs produced by evaluating the summation of the Riemann zeta function's terms for various values of the argument. The summation of the first n terms was graphed as a point in the complex plane, and n went from one to large integer. Computer calculations were used to produce images for investigation. The zeta function, for the purposes of the experiment, was defined as:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \ .$$

where s = a + bi, and a and b are real numbers and i is the imaginary number, $\sqrt{-1}$.

The experiment found remarkable patterns and structure in the graphs for large values of b, tending to create spirals and curves.

Introduction

The zeta function is a unique special function of mathematics that arises from definite integration. It has deep connections to prime number theory and is related to many important conjectures. The study of the function was founded by Bernhard Riemann's 1859 eight page paper, "On the Number of Primes Less than a Given Magnitude" (Wolfram). The paper's roots lie in Euclid's anaylsis of prime numbers in the eighteenth century (Edwards, 2001).

There are many different ways of expressing the zeta function. It can be written as a definite integral, as a functional equation, in terms of other special functions, or as several different series. Although the function in (1) does not converge for nonreal values, the zeta function can be rewritten differently as an analytic continuation that will converge for complex values (Wolfram).

The Riemann Hypothesis states that all of the nontrivial zeroes of the function have real part equal to ½ (the "critical line" in the complex plane). Althought the hypothesis has not yet been proven or disproven, it has been verified for the first 10^{11} zeroes (Wolfram).

When Lon Radin (the father of the author) was seeking to develope a method for finding zeroes of the zeta function, he noticed that the summation began to appear to converge on a value, when it would later jump to another value. His research was later expanded to this experiment.

It was hypothesized that the graphs would show significant and definite structure.

Methods and Materials

Images were produced using Borland Turbo C++ 3.0, an antique DOS compiler. Images were processed using IrfanView, a Windows freeware graphics tool made by Irfan Skiljan. All computation was performed on a 750Mhz PC running Windows ME.

The program used basic mathematical functions included in Turbo C++ to sum a number of the terms of the series. The image was stored in a 320 kb buffer as an array of one bit pixels. A function then copied the header data from a template image file and wrote the image data from the buffer to a file. Images were made in th PCX format, which uses a simple form of run length encoding.

Adjustments to program parameters were made throughout the experiment to ensure the graph was within the viewing area and of appropriate magnification. Graphs were made with various different imaginary values, some of which were zeroes of the function, and the real part being ½.

A copy of the program and other similar programs is available at: http://www.geocities.com/kasplurpo/zeta.htm

Results

(1) It was observed that small values of b (of the order 10^2) generated a spiral that expanded outward with a hole in the middle.

(2) It was observed that for large values of b (of the order 10^5) there were regions of sparse dots that gradually formed a series of spiral shapes as more terms were added. These spirals grew consecutively in size, but had holes in the center that diminished consecutively in size (see Figure 1, page 11).

(3) It was also observed that larger values of b (of the order 10^{11}) the spirals were significantly more numerous and smaller in size comared to the whole of the graph. The series of spirals sometimes traced a distinct curved path (see Figure 2, page 12). This curve, in some cases, traced more than one complete revolution around a circular path (see Figure 3, page 12).

Discussion

(1) The presence of a single outward spiral can be mathematically explained. Because the zeta function in the form used in the experiment does not coverge, its graph will continually expand as the number of terms summed approaches infinity, rather than converge on a point. This will be further discussed in (2).

(2) The presence of spirals can be mathematically explained. Using the Euler Formula ($e^{ix} = \cos(x) + i*\sin(x)$) the function can be rewritten as follows, where s = a + bi:

$z(s) = \sum k^{-s}$	Given
$z(s) = \sum k^{-a-bi}$	Properties of Exponents
$z(s) = \sum k^{-a} k^{-bi}$	Properties of Exponents
$z(s) = \sum k^{-a} e^{-bi \ln(k)}$	Properties of Exponents
$z(s) = \sum k^{-a} [\cos(-b \ln(k)+i\sin(-b \ln(k))]$	Euler's Formula

Now each term can be interpreted as a vector of angle -b ln(k) and magnitude k^{-a}. The change in angle, $d\theta$, between two consecutive vectors can be calculated using calculus:

$\theta = -b \ln(k)$	Given
$d\theta = D_k(-b \ln(k))$	Definition of a Derivative
$d\theta = -b\ D_k(\ln(k))$	Property of Derivaties
$d\theta = -b/k$	Derivative of a Natural Logarithm

The reference angle of $d\theta$ will go from 0 to 2π repeatedly as k increases. That means that the rate at which the angle of the curve changes will vary cyclicly. Where $d\theta = 2n\pi$ and n is an integer, the curve will be straight, because consecutive vectors will point in the same direction. $d\theta$ will then increase, making a tighter and tighter turn until $d\theta = (2n+1)\pi$, where consecutive vectors will point in opposite direction, at the center of the spiral. $d\theta$ will continue to increase, spiraling outward, until $d\theta = (2n+2)\pi = 2n\pi$, where the process repeats.

The number of spirals will be equal to the number of values of k for which dθ is divisible by 2π. This is equal to b/(2π). Because the earlier spirals (where k is small) are closer together and each contain fewer points, they are less recognizable and resemble randomly scattered dots. Later spirals (where k is large) are much more easily identifiable. This is well illustrated in Figure 1.

As k approaches infinity, dθ and the vector magnitude (k^{-a}) approach zero. This indicates that as k^{-s} is summed indefinitely, the graph will spiral outwards, as observed in (1).

(3) The larger curves produced by many smaller spirals of the type observed in (2) are not accounted for. The implication is that the change in angle and/or the absolute angle of the straight sections between spirals is not arbitrary. Evidently, there is a strong tendency towards structure. There is presumbly an explaination, but that explaination exceeds the knowledge of the author.

From the observations, it was evident that the graphs had significant and definite structure, affirming the hypothesis.

Acknowledgements

Recognition is deserved for those who contributed valuable resources:

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Lon Radin

Literature Cited

Edwards, HM. Riemann's Zeta Function. New York: Dover Publications, 2001.

http://mathworld.wolfram.com/RiemannZetaFunction.html, Riemann Zeta Function, Jonathan Sondow

Tables and Figures

Graphs of the Summations of z(s) for various values of s:

Figure 1: s = ½ + 936391.0i (a zero), computed to $2*10^5$ terms

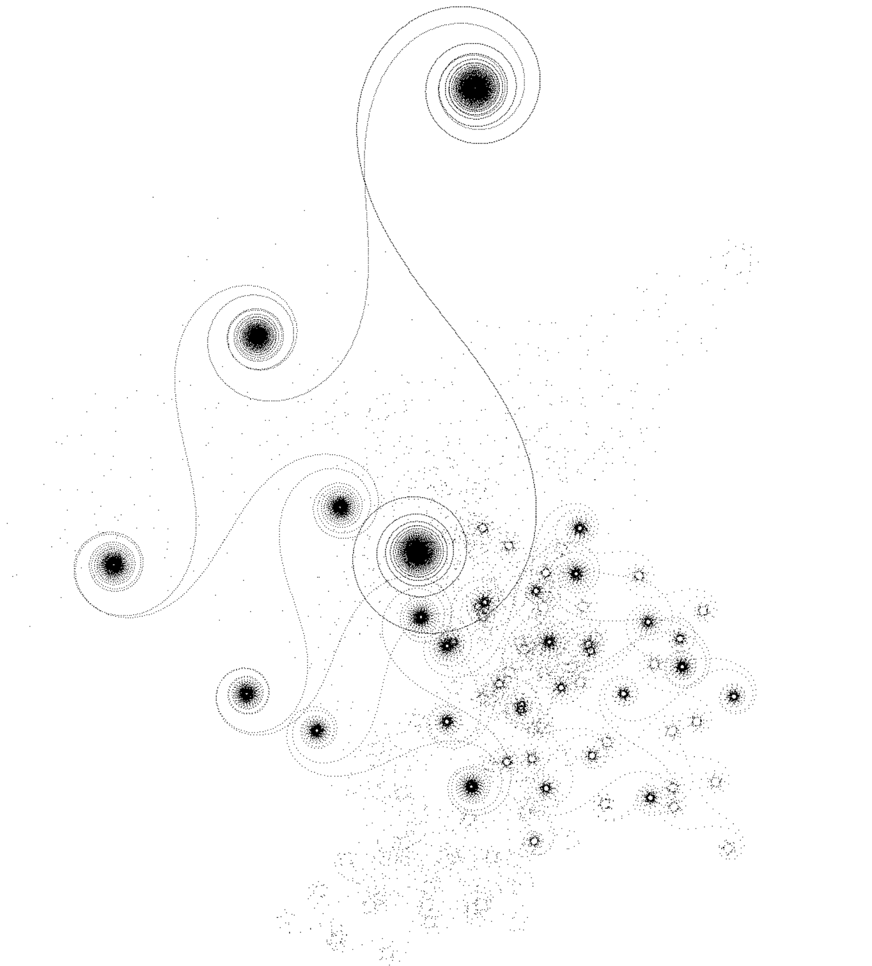


Figure 2: s = ½ + 267,653,395,648.848i (zero number $10^{12}+1$), computed to 10^7 terms

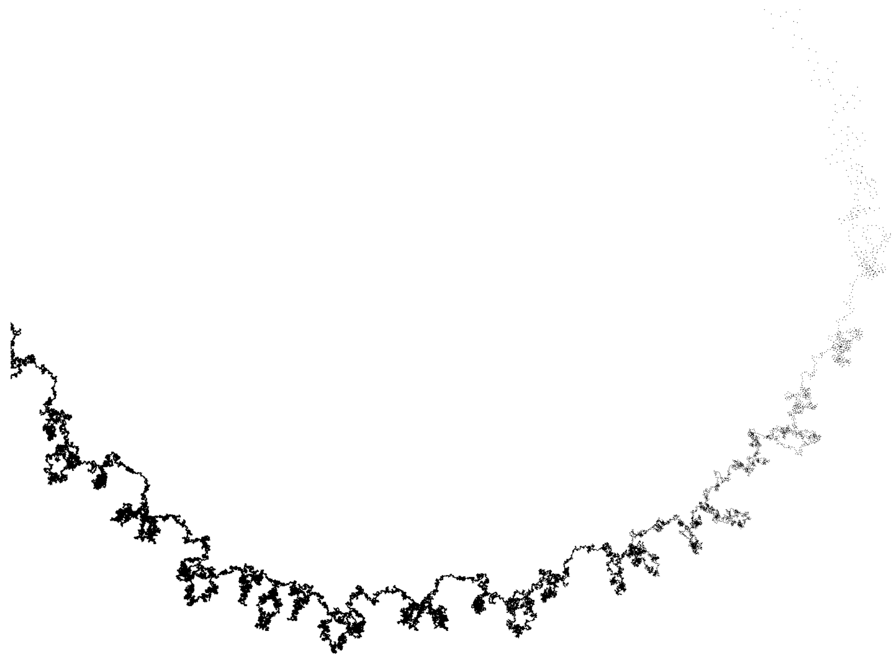
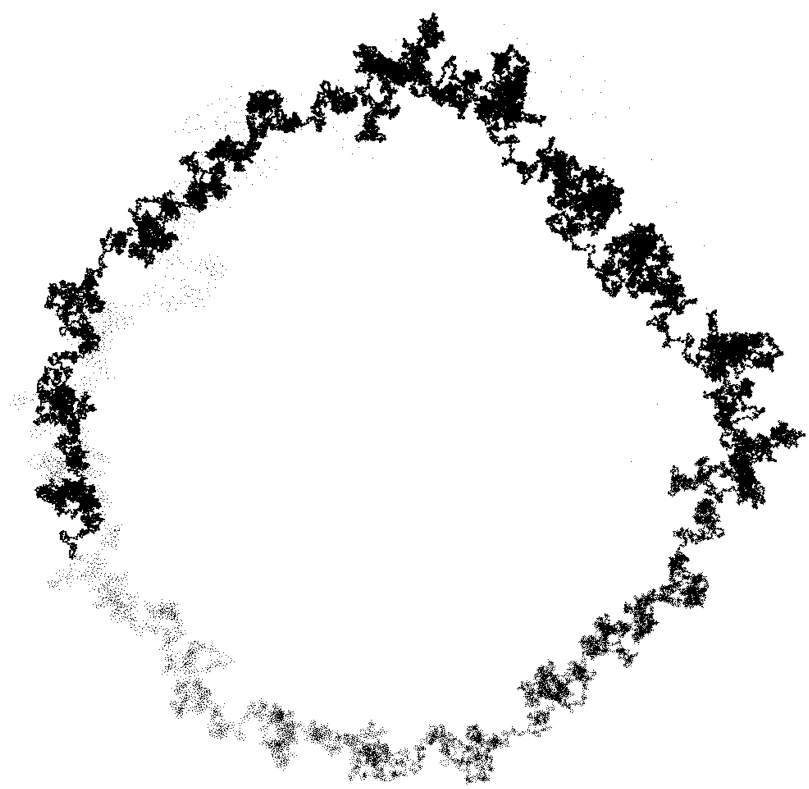


Figure 3: s = ½ +210,000,000,000.000 (not a zero), computed to 10^7 terms



Not all images are shown here due to space limitations. Additional images are available at: http://www.geocities.com/kasplurpo/zeta.htm