Spread option pricing tool.

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Abstract

This article contains calculation of premium and Greeks of a spread option on two correlated assets under assumptions of the Black model.

1 Spread option.

The spread option pays at the maturity T

$$\left(F_{1,T}-F_{2,T}-K\right)_{+}$$

where the $F_{i,T}$ are some forward prices at T and the K is a strike (real valued parameter). We assume the geometric Brownian set up

$$\frac{dF_{1,t}}{F_{1,t}}=\sigma_1 dW_{1,t},$$

$$\frac{dF_{2,t}}{F_{2,t}}=\rho\sigma_2 dW_{1,t}+\sqrt{1-\rho^2}\sigma_2 dW_{2,t},$$

where the ρ is the correlation of the prices and σ_i, $i=1,2$ are volatilities of the assets (positive real valued parameters). In light of the computations leading to the Black-Scholes formula (see the Notes) we write the price of the option at time $t=0$ as

$$V=e^{-rT}E\left(F_{1,T}-F_{2,T}-K\right)_{+}=$$

$$=e^{-rT}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-F_{2,0}e^{-\frac{1}{2}\sigma_2^2T+\sigma_2\left(\rho x+\sqrt{1-\rho^2}y\right)\sqrt{T}}-K\right)_{+}$$

$$\frac{1}{2\pi}e^{-\frac{1}{2}\left(x^2+y^2\right)}dxdy.$$

Denote $H\left(x,y\right)$ the expression inside the $\left(...\right)_{+}$ term.

1.1 The strike K is positive.

Observe that

$$(F_1 - F_2 - K)_+ + (F_1 - F_2 - K)_- = F_1 - F_2 - K,$$

$$(F_1 - F_2 - K)_- = (-(F_2 - F_1 + K))_- = -(F_2 - F_1 + K)_+ .$$

Hence,

$$E(F_{1,T} - F_{2,T} - K)_+ - E(F_{2,T} - F_{1,T} + K)_+ = E(F_{1,T} - F_{2,T} - K) =$$

$$= F_{1,0} - F_{2,0} - K.$$

Therefore, the spread with positive strike may be calculated via the spread with negative strike according to the formula

$$E(F_{1,T} - F_{2,T} - K)_+ = E(F_{2,T} - F_{1,T} + K)_+ + F_{1,0} - F_{2,0} - K.$$

1.2 The strike K is not positive.

Note, that the equation $H(x,y) = 0$ has a solution $y = h(x)$ for any value x such that $F_{1,0}e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 x\sqrt{T}} - K > 0$. If K is not positive then this is always so. We may produce an explicit expression for the $h(x)$. Indeed,

$$F_{1,0}e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 x\sqrt{T}} - F_{2,0}e^{-\frac{1}{2}\sigma_2^2 T + \sigma_2\left(\rho x + \sqrt{1-\rho^2}h(x)\right)\sqrt{T}} - K = 0$$

transforms to

$$h(x) = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\log\left(\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 x\sqrt{T}} - K}{F_{2,0}}\right) + \frac{1}{2}\sigma_2^2 T}{\sigma_2\sqrt{T}} - \rho x\right).$$

We split the integral as follows

$$V = e^{-rT}(I_1 - I_2 - I_3),$$

$$I_1 = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{h(x)} dy\, F_{1,0}e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 x\sqrt{T}} \frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2+y^2\right)},$$

$$I_2 = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{h(x)} dy\, F_{2,0}e^{-\frac{1}{2}\sigma_2^2 T + \sigma_2\left(\rho x + \sqrt{1-\rho^2}y\right)\sqrt{T}} \frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2+y^2\right)},$$

$$I_3 = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{h(x)} dy\, K \frac{1}{2\pi} e^{-\frac{1}{2}\left(x^2+y^2\right)}.$$

Next, we transform each of the integrals I_i using the error function $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}s^2} ds$:

$$I_1 = F_{1,0} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 - \frac{1}{2}\sigma_1^2 T + \sigma_1 x \sqrt{T}} dx \int_{-\infty}^{h(x)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy =$$

$$= F_{1,0} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x - \sigma_1 \sqrt{T}\right)^2} N\left(h\left(x\right)\right)\, dx\ =$$

$$= F_{1,0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}s^2} N\left(h\left(s + \sigma_1 \sqrt{T}\right)\right)\, ds,$$

$$I_2 = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2 + \sigma_2 \rho x \sqrt{T} - \frac{1}{2}\sigma_2^2 \rho^2 T} dx \cdot$$

$$\cdot \int_{-\infty}^{h(x)} F_{2,0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2 + \sigma_2 \sqrt{1-\rho^2} y \sqrt{T} - \frac{1}{2}\sigma_2^2 \left(1-\rho^2\right) T} dy =$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(x - \sigma_2 \rho \sqrt{T}\right)^2} dx \int_{-\infty}^{h(x)} F_{2,0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y - \sigma_2 \sqrt{1-\rho^2} \sqrt{T}\right)^2} dy =$$

$$= F_{2,0} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}v^2} dv \int_{-\infty}^{h\left(v + \sigma_2 \rho \sqrt{T}\right) - \sigma_2 \sqrt{1-\rho^2} \sqrt{T}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2} du =$$

$$= F_{2,0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}v^2}\, N\left(h\left(v + \sigma_2 \rho \sqrt{T}\right) - \sigma_2 \sqrt{1-\rho^2} \sqrt{T}\right) dv,$$

$$I_3 = K \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\, N\left(h\left(x\right)\right)\, dx.$$

Therefore,

$$V = e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\, F\left(x\right) dx,$$

$$F\left(x\right) = F_{1,0} N\left(h\left(x + \sigma_1 \sqrt{T}\right)\right) - K\, N\left(h\left(x\right)\right)$$

$$- F_{2,0} N\left(h\left(x + \sigma_2 \rho \sqrt{T}\right) - \sigma_2 \sqrt{1-\rho^2} \sqrt{T}\right).$$

1.3 Put.

Put is the derivative with the payoff

$$\left(K - F_1 + F_2\right)_+ .$$

Hence, this is a call with a swap of input data $K \to -K$ and $F_1 \to F_2$.

1.4 Calculation for call-put and all strikes.

Denote $V_0\left(K, F_{1,0}, F_{2,0}, \sigma_1, \sigma_2\right)$ the pricing formula for a call with non-positive strike and $V\left(K, F_{1,0}, F_{2,0}, \sigma_1, \sigma_2, \cdot\right)$ the pricing procedure for any strike, call or put:

$$V\left(K \leq 0, F_{1,0}, F_{2,0}, \sigma_1, \sigma_2, \text{call}\right) = V_0\left(K, F_{1,0}, F_{2,0}, \sigma_1, \sigma_2\right).$$

We have

$$\begin{gathered} V\left(K > 0, F_{1,0}, F_{2,0}, \sigma_1, \sigma_2, \text{call}\right) = \\ = V_0\left(-K, F_{2,0}, F_{1,0}, \sigma_2, \sigma_1\right) + e^{-rT}\left(F_{1,0} - F_{2,0} - K\right), \\ V\left(K \leq 0, F_{1,0}, F_{2,0}, \sigma_1, \sigma_2, \text{put}\right) = V\left(-K > 0, F_{2,0}, F_{1,0}, \sigma_2, \sigma_1, \text{call}\right) = \\ = V_0\left(K, F_{1,0}, F_{2,0}, \sigma_1, \sigma_2\right) + e^{-rT}\left(F_{2,0} - F_{1,0} + K\right), \\ V\left(K > 0, F_{1,0}, F_{2,0}, \sigma_1, \sigma_2, \text{put}\right) = V\left(-K \leq 0, F_{2,0}, F_{1,0}, \sigma_2, \sigma_1, \text{call}\right) = \\ = V_0\left(-K, F_{2,0}, F_{1,0}, \sigma_2, \sigma_1\right). \end{gathered}$$

2 The Gauss Hermite procedure.

The original formula ("Handbook of mathematical functions" M. Abramowitz and I.Stegun, Dover publications. Pages 890 and 924.) reads

$$\int_{-\infty}^{+\infty} e^{-x^2} f(x)\, dx = \sum_{i=1}^{n} w_i f\left(x_i\right) + R_n,$$

$$R_n = \frac{n!\sqrt{\pi}}{2^n (2n)!} f^{(2n)}(\xi), \; -\infty < \xi < +\infty,$$

where the coefficients are

$n = 20$

$\pm x_i$	w_i
0.2453407083009	4.622436696006e-1
0.7374737285454	2.866755053628e-1
1.2340762153953	1.090172060200e-1
1.7385377121166	2.481052088746e-2
2.2549740020893	3.243773342238e-3
2.7888060584281	2.283386360163e-4
3.3478545673832	7.802556478532e-6
3.9447640401156	1.086069370769e-7
4.6036824495507	4.399340992273e-10
5.3874808900112	2.229393645534e-13

$n = 12$

$\pm x_i$	w_i
0.314240376254359	5.701352362625e-1
0.947788391240164	2.604923102642e-1
1.597682635152605	5.160798561588e-2
2.279507080501060	3.905390584629e-3
3.020637025120890	8.573687043588e-5
3.889724897869782	2.658551684356e-7

2.1 Adaptation of the Gauss-Hermite procedure.

We are interested in evaluation of $\int_{-\infty}^{+\infty} \frac{e^{-\frac{x}{2}^{2}}}{\sqrt{2\pi}} f(x)\,dx$. We have

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{x}{2}^{2}}}{\sqrt{2\pi}} f(x)\,dx \underset{\frac{x}{\sqrt{2}}=y}{=} \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{\sqrt{2\pi}} f\left(y\sqrt{2}\right)\sqrt{2}dy =$$

$$= \int_{-\infty}^{+\infty} \frac{e^{-y^2}}{\sqrt{\pi}} f\left(y\sqrt{2}\right) dy = \sum_{i=1}^{n} w_i \frac{f\left(x_i\sqrt{2}\right)}{\sqrt{\pi}} + R_n =$$

$$= \sum_{i=1}^{n} u_i f(y_i) + R_n,$$

$$u_i = \frac{w_i}{\sqrt{\pi}}, \quad y_i = x_i\sqrt{2}.$$

$n = 20$

$\pm y_i$	u_i
0.346964157081354	2.60793063449549E-01
1.04294534880276	1.61739333983981E-01
1.74524732081409	6.15063720639637E-02
2.45866361117239	1.39978374470990E-02
3.18901481655342	1.83010313108057E-03
3.94396735065727	1.28826279961899E-04
4.73458133404603	4.40212109023082E-06
5.57873880589317	6.12749025998136E-08
6.51059015701359	2.48206236231508E-10
7.61904854167971	1.25780067243784E-13

$n = 12$

$\pm y_i$	u_i
0.444403001944139	3.21664361512842E-01
1.34037519715162	1.46967048045352E-01
2.25946445100079	2.91166879123619E-02
3.22370982877010	2.20338068753316E-03
4.27182584793228	4.83718492259071E-05
5.50090170446775	1.49992716763700E-07

2.2 Testing the Gauss-Hermite procedure.

The procedure was tested for the integrals

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{x}{2}^2}}{\sqrt{2\pi}} dx = 1,$$

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{x}{2}^2}}{\sqrt{2\pi}} \sin(x)\, dx = 0,$$

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{x}{2}^2}}{\sqrt{2\pi}} \cos(x)\, dx = \frac{1}{\sqrt{e}},$$

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{x}{2}^2}}{\sqrt{2\pi}} \exp(x)\, dx = \sqrt{e}.$$

The $n = 12$ procedure evaluates the above integrals with precision 10^{-13}.

The integral

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{x}{2}^2}}{\sqrt{2\pi}} x^{10} dx = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 945$$

evaluates with precision 10^{-6}.

3 Greeks.

The premium is given by

$$V = e^{-rT} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}x^2}\, F(x)\, dx,$$

$$\begin{aligned} F(x) = {} & F_{1,0} N\left(h\left(x + \sigma_1\sqrt{T}\right)\right) - K\, N(h(x)) \\ & - F_{2,0} N\left(h\left(x + \sigma_2\rho\sqrt{T}\right) - \sigma_2\sqrt{1-\rho^2}\sqrt{T}\right). \end{aligned}$$

Hence, we have to calculate the derivatives of the function $F(x)$. Let us denote $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.

$$\frac{\partial F(x)}{\partial F_{1,0}} = N\left(h\left(x + \sigma_1\sqrt{T}\right)\right) + F_{1,0}\left(p(h(s))\frac{\partial h(s)}{\partial F_{1,0}}\right)_{s=x+\sigma_1\sqrt{T}}$$

$$-K\, p(h(x))\frac{\partial h(x)}{\partial F_{1,0}} - F_{2,0}\left(p\left(h(s) - \sigma_2\sqrt{1-\rho^2}\sqrt{T}\right)\frac{\partial h(s)}{\partial F_{1,0}}\right)_{s=x+\sigma_2\rho\sqrt{T}},$$

$$\frac{\partial F(x)}{\partial F_{2,0}} = F_{1,0}\left(p(h(s))\frac{\partial h(s)}{\partial F_{2,0}}\right)_{s=x+\sigma_1\sqrt{T}}$$

$$-K\,p\left(h\left(x\right)\right)\frac{\partial h\left(x\right)}{\partial F_{2,0}}-F_{2,0}\left(p\left(h\left(s\right)-\sigma_2\sqrt{1-\rho^2}\sqrt{T}\right)\frac{\partial h\left(s\right)}{\partial F_{2,0}}\right)_{s=x+\sigma_2\rho\sqrt{T}}$$

$$-N\left(h\left(x+\sigma_2\rho\sqrt{T}\right)-\sigma_2\sqrt{1-\rho^2}\sqrt{T}\right).$$

$$\left(\frac{\partial F\left(x\right)}{\partial u}\right)_{u=\sigma_1,\sigma_2,\rho,T}=F_{1,0}\left(p\left(h\left(s\right)\right)\left(\frac{\partial h\left(s\right)}{\partial u}+\frac{\partial h\left(s\right)}{\partial s}\frac{\partial s}{\partial u}\right)\right)_{s=x+\sigma_1\sqrt{T}}$$

$$-K\,p\left(h\left(x\right)\right)\frac{\partial h\left(x\right)}{\partial u}$$

$$-F_{2,0}\{p\left(h\left(s\right)-\sigma_2\sqrt{1-\rho^2}\sqrt{T}\right)\cdot$$

$$\cdot\left(\frac{\partial h\left(s\right)}{\partial u}+\frac{\partial h\left(s\right)}{\partial s}\frac{\partial s}{\partial u}-\frac{\partial\left(\sigma_2\sqrt{1-\rho^2}\sqrt{T}\right)}{\partial u}\right)\}_{s=x+\sigma_2\rho\sqrt{T}}.$$

It remains to calculate derivatives of the function

$$h\left(x\right)=\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\log\left(\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}{F_{2,0}}\right)+\frac{1}{2}\sigma_2^2T}{\sigma_2\sqrt{T}}-\rho x\right).$$

We have

$$\frac{\partial h\left(x\right)}{\partial x}=\frac{1}{\sqrt{1-\rho^2}}\cdot$$

$$\left(\frac{1}{\sigma_2\sqrt{T}}\frac{F_{2,0}}{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}\sigma_1\sqrt{T}}{F_{2,0}}-\rho\right)=$$

$$=\frac{1}{\sqrt{1-\rho^2}}\left(\frac{\sigma_1}{\sigma_2}\frac{1}{1-\frac{K}{F_{1,0}}e^{\frac{1}{2}\sigma_1^2T-\sigma_1x\sqrt{T}}}-\rho\right).$$

$$\frac{\partial h\left(x\right)}{\partial F_{1,0}}=\frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sigma_2\sqrt{T}}\frac{F_{2,0}}{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}=$$

$$=\frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sigma_2\sqrt{T}}\frac{F_{2,0}}{F_{1,0}}\frac{1}{1-\frac{K}{F_{1,0}}e^{\frac{1}{2}\sigma_1^2T-\sigma_1x\sqrt{T}}}.$$

$$\frac{\partial h\left(x\right)}{\partial F_{2,0}}=\frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sigma_2\sqrt{T}}\frac{F_{2,0}}{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}\cdot$$

$$\cdot\left(-1\right)\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}{\left(F_{2,0}\right)^2}=$$

$$= -\frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sigma_2\sqrt{T}}\frac{1}{F_{2,0}}.$$

$$\frac{\partial h\left(x\right)}{\partial\sigma_1} = \frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sigma_2\sqrt{T}}\frac{F_{2,0}}{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}\cdot$$

$$\cdot\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}\left(-\sigma_1T+x\sqrt{T}\right)}{F_{2,0}} =$$

$$= \frac{1}{\sqrt{1-\rho^2}}\frac{1}{\sigma_2\sqrt{T}}\frac{-\sigma_1T+x\sqrt{T}}{1-\frac{K}{F_{1,0}}e^{\frac{1}{2}\sigma_1^2T-\sigma_1x\sqrt{T}}}.$$

$$\frac{\partial h\left(x\right)}{\partial\sigma_2} = \frac{1}{\sqrt{1-\rho^2}}\frac{\sigma_2T\,\sigma_2\sqrt{T}-\left(\log\left(\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}{F_{2,0}}\right)+\frac{1}{2}\sigma_2^2T\right)\sqrt{T}}{\sigma_2^2T} =$$

$$= \frac{1}{\sqrt{1-\rho^2}}\left(\frac{1}{2}-\frac{1}{\sigma_2^2T}\log\left(\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}{F_{2,0}}\right)\right)\sqrt{T}.$$

$$\frac{\partial h\left(x\right)}{\partial\rho} = \frac{-\frac{1}{2}}{\left(1-\rho^2\right)^{\frac{3}{2}}}\left(\frac{\log\left(\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}{F_{2,0}}\right)+\frac{1}{2}\sigma_2^2T}{\sigma_2\sqrt{T}}-\rho x\right)-\frac{x}{\sqrt{1-\rho^2}}.$$

$$\frac{\partial h\left(x\right)}{\partial T} = \frac{1}{\sigma_2\sqrt{1-\rho^2}}\frac{\partial}{\partial T}\left(\frac{1}{\sqrt{T}}\log\left(\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}{F_{2,0}}\right)+\frac{1}{2}\sigma_2^2\sqrt{T}\right) =$$

$$= \frac{1}{\sigma_2\sqrt{1-\rho^2}}\{\frac{-\frac{1}{2}}{T^{\frac{3}{2}}}\log\left(\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}{F_{2,0}}\right)+$$

$$+\frac{1}{\sqrt{T}}\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}\frac{1}{2}\left(-\sigma_1^2+\frac{\sigma_1x}{\sqrt{T}}\right)}{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}+\frac{1}{2}\sigma_2^2\frac{1}{2\sqrt{T}}\} =$$

$$= \frac{1}{2\sqrt{T}\,\sigma_2\sqrt{1-\rho^2}}\left(-\frac{\log\left(\frac{F_{1,0}e^{-\frac{1}{2}\sigma_1^2T+\sigma_1x\sqrt{T}}-K}{F_{2,0}}\right)}{T}+\frac{\sigma_1\left(-\sigma_1+\frac{x}{\sqrt{T}}\right)}{1-\frac{K}{F_{1,0}}e^{\frac{1}{2}\sigma_1^2T-\sigma_1x\sqrt{T}}}+\frac{\sigma_2^2}{2}\right).$$

4 Optimization of calculation.

If the above formulas are substituted directly to the Gauss-Hermite integrator then many complex expressions will be evaluated repeatedly. The goal of this section is to put final formulas in a form that avoids repeated evaluations.

Let us introduce the following notations

$$y_i^{(1)} = -y_i,\ i = 1, ..., \frac{n}{2};\quad y_i^{(1)} = y_{i-\frac{n}{2}},\ \ i = \frac{n}{2} + 1, ..., n;$$

$$y_i^{(2)} = y_i^{(1)} + \sigma_1 \sqrt{T},$$
$$y_i^{(3)} = y_i^{(1)} + \sigma_2 \rho \sqrt{T},$$

where the y_i are the values from description of the Gauss-Hermite integration procedure,

$$C_1 = \sigma_1^2,\ \ C_2 = \sigma_2^2,$$
$$C_3 = \sigma_1^2 T = C_1 T,\ \ C_4 = \sigma_2^2 T = C_2 T,$$
$$C_5 = \frac{1}{2}\sigma_1^2 T = \frac{1}{2} C_3,\ \ C_6 = \frac{1}{2}\sigma_2^2 T = \frac{1}{2} C_4,$$
$$C_7 = \sqrt{T},\ \ C_8 = \sigma_1 \sqrt{T} = \sigma_1 C_7,\ \ C_9 = \sigma_2 \sqrt{T} = \sigma_2 C_7,$$
$$C_{10} = \frac{K}{F_{1,0}},\ \ C_{11} = \frac{K}{F_{2,0}},\ \ C_{12} = \frac{F_{1,0}}{F_{2,0}},$$
$$C_{13} = \sqrt{1-\rho^2}, C_{14} = \frac{\sigma_1}{\sigma_2},$$
$$\alpha_i^{(j)} = e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 y_i^{(j)} \sqrt{T}} = e^{C_8 y_i^{(j)} - C_5},\ j = 1, 2, 3,$$
$$\beta_i^{(j)} = \frac{1}{1 - \frac{K}{F_{1,0}} e^{\frac{1}{2}\sigma_1^2 T - \sigma_1 y_i^{(j)} \sqrt{T}}} = \frac{1}{1 - \frac{C_{10}}{\alpha_i^{(j)}}},$$
$$\gamma_i^{(j)} = \log\left(\frac{F_{1,0} e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 x \sqrt{T}} - K}{F_{2,0}}\right) = \log\left(C_{12}\alpha_i^{(j)} - C_{11}\right),$$
$$\delta_i^{(j)} = h\left(y_i^{(j)}\right) = \frac{\frac{\gamma_i^{(j)} + C_6}{C_9} - \rho y_i^{(j)}}{C_{13}}$$
$$A_i^{(j)} = \left(\frac{\partial h\,(x)}{\partial x}\right)_{x = y_i^{(j)}} = \frac{C_{14}\beta_i^{(j)} - \rho}{C_{13}},$$
$$B_i^{(j)} = \frac{\partial h\left(y_i^{(j)}\right)}{\partial F_{1,0}} = \frac{\beta_i^{(j)}}{C_{13} C_9 C_{12}},$$

$$C_{15} = \frac{\partial h\left(y_i^{(j)}\right)}{\partial F_{2,0}} = \frac{-1}{C_{13}C_9F_{2,0}},$$

$$D_i^{(j)} = \frac{\partial h\left(y_i^{(j)}\right)}{\partial \sigma_1} = \frac{y_i^{(j)}\sqrt{T} - \sigma_1 T}{C_{13}C_9}\beta_i^{(j)},$$

$$E_i^{(j)} = \frac{\partial h\left(y_i^{(j)}\right)}{\partial \sigma_2} = \left(\frac{1}{2} - \frac{\gamma_i^{(j)}}{C_4}\right)\frac{C_7}{C_{13}},$$

$$F_i^{(j)} = \frac{\partial h\left(y_i^{(j)}\right)}{\partial \rho} = \frac{-\frac{1}{2}}{1-\rho^2}\delta_i^{(j)} - \frac{y_i^{(j)}}{C_{13}},$$

$$G_i^{(j)} = \frac{\partial h\left(y_i^{(j)}\right)}{\partial T} = \frac{\frac{1}{2}}{C_7\sigma_2C_{13}}\left(\left(\frac{y_i^{(j)}}{C_7}\sigma_1 - C_1\right)\beta_i^{(j)} - \frac{\gamma_i^{(j)}}{T} + \frac{1}{2}C_2\right),$$

$$N_i^{(1)} = N\left(h\left(y_i^{(1)}\right)\right) = N\left(\delta_i^{(1)}\right),$$

$$N_i^{(2)} = N\left(h\left(y_i^{(2)}\right)\right) = N\left(\delta_i^{(2)}\right),$$

$$N_i^{(3)} = N\left(h\left(y_i^{(3)}\right) - C_{12}C_9\right) = N\left(\delta_i^{(3)} - C_{13}C_9\right),$$

$$P_i^{(1)} = p\left(h\left(y_i^{(1)}\right)\right) = p\left(\delta_i^{(1)}\right),$$

$$P_i^{(2)} = p\left(h\left(y_i^{(2)}\right)\right) = p\left(\delta_i^{(2)}\right),$$

$$P_i^{(3)} = p\left(h\left(y_i^{(3)}\right) - C_{12}C_9\right) = p\left(\delta_i^{(3)} - C_{13}C_9\right),$$

$$F\left(y_i\right) = F_{1,0}N_i^{(2)} - K\,N_i^{(1)} - F_{2,0}N_i^{(3)},$$

$$\frac{\partial F\left(y_i\right)}{\partial F_{1,0}} = N_i^{(2)} + F_{1,0}P_i^{(2)}B_i^{(2)} - K\,P_i^{(1)}B_i^{(1)} - F_{2,0}P_i^{(3)}B_i^{(3)},$$

$$\frac{\partial F\left(y_i\right)}{\partial F_{2,0}} = \left(F_{1,0}P_i^{(2)} - K\,P_i^{(1)} - F_{2,0}P_i^{(3)}\right)C_{15} - N_i^{(3)},$$

$$\frac{\partial F\left(y_i\right)}{\partial \sigma_1} = F_{1,0}P_i^{(2)}\left(D_i^{(2)} + A_i^{(2)}C_7\right) - KP_i^{(1)}D_i^{(1)} - F_{2,0}P_i^{(3)}D_i^{(3)},$$

$$\frac{\partial F\left(y_i\right)}{\partial \sigma_2} = F_{1,0}P_i^{(2)}E_i^{(2)} - KP_i^{(1)}E_i^{(1)} - F_{2,0}P_i^{(3)}\left(E_i^{(3)} + A_i^{(3)}\rho C_7 - C_7C_{13}\right),$$

$$\frac{\partial F\left(y_i\right)}{\partial \rho} = F_{1,0}P_i^{(2)}F_i^{(2)} - KP_i^{(1)}F_i^{(1)} - F_{2,0}P_i^{(3)}\left(F_i^{(3)} + A_i^{(3)}C_9 + C_9\frac{\rho}{C_{13}}\right),$$

$$\frac{\partial F\left(y_i\right)}{\partial T} = F_{1,0}P_i^{(2)}\left(G_i^{(2)} + A_i^{(2)}\frac{\sigma_1}{2C_7}\right) - KP_i^{(1)}G_i^{(1)}$$

$$-F_{2,0}P_i^{(3)}\left(G_i^{(3)} + A_i^{(3)}\frac{\sigma_2\rho}{2C_7} - \frac{\sigma_2 C_{13}}{2C_7}\right).$$

Finally, the premium V and Greeks $\left(\frac{\partial V}{\partial v}\right)_{v=F_{1,0},F_{2,0},\sigma_1,\sigma_2,\rho}$, $\frac{\partial V}{\partial T}$, $\frac{\partial V}{\partial r}$ are calculated by the following recipes:

$$V = e^{-rT}\sum_{i=1}^{\frac{n}{2}} u_i\left(F\left(y_i\right) + F\left(y_{i+\frac{n}{2}}\right)\right),$$

$$\left(\frac{\partial V}{\partial v}\right)_{v=F_{1,0},F_{2,0},\sigma_1,\sigma_2,\rho} = e^{-rT}\sum_{i=1}^{n} u_i\left(\frac{\partial F\left(y_i\right)}{\partial v} + \frac{\partial F\left(y_{i+\frac{n}{2}}\right)}{\partial v}\right),$$

$$\frac{\partial V}{\partial T} = e^{-rT}\sum_{i=1}^{n} u_i\left(\frac{\partial F\left(y_i\right)}{\partial T} + \frac{\partial F\left(y_{i+\frac{n}{2}}\right)}{\partial T}\right) - rV,$$

$$\frac{\partial V}{\partial r} = -TV.$$