Spreading Control Laws : Algorithms and Applications [a]

to : **Jean-Pierre Aubin** for his 65th Birthday

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INTRODUCTION

Spreading concept has been introduced in order to model and control expansion phenomena that arise in spatially distributed processes such as :

- Pollution in air quality
- Desertification in vegetation dynamics
- Cancer cells in biomedicine
- Biological control and invasion theory

Main Works :

94 Introduction in the litterature with A. El Jai in MCM

96 The case of Transport systems in IJSS

97 Spreading control using Optimal Control Techniques in the case of Linear systems, in IJC

98 A set-valued approach communicated in SIAM Conf. on Cont.

00 By monotonocity w. r. to a preorder in SCL

02 feedback spreading control under speed constraints, in SICON

PRESENTATION

It consists of :

a system : $\frac{\partial z}{\partial t} + Az = \varphi(z, v)$ **in** $\Omega \times (0, \infty[,$

- $\Omega \subset \mathbb{R}^n (n = 1, 2$ **or** $3)$
- $-A$ **generates a** $\mathcal{C}_0$ **semigroup** $S(\cdot) on £ Z = L^2(\Omega)$
- $z(\cdot, 0) = z_0 \in \mathrm{dom}(A)$
- $\varphi : \mathcal{S} \times V \mapsto Z$ $(V$ **Hilbert sp. of control,** $\mathcal{S} \subset Z)$

a map to be spread : $\omega : \mathcal{S} \subset Z \to 2^\Omega$

Definitions :

- $v = v(\cdot)$ **a spreading control w. r. to** ω **if exists a solution** $\bar{z}$ **such that :** $(\omega(\bar{z}(t))_t$ **increases**
- $\varsigma : \mathcal{S} \to V$ **is said a Feedback Spreading Control law w. r. to** ω **if,** [**for all** z_0 **in** $\mathcal{S}$, $v = \varsigma(z)$ **is a spreading control**]

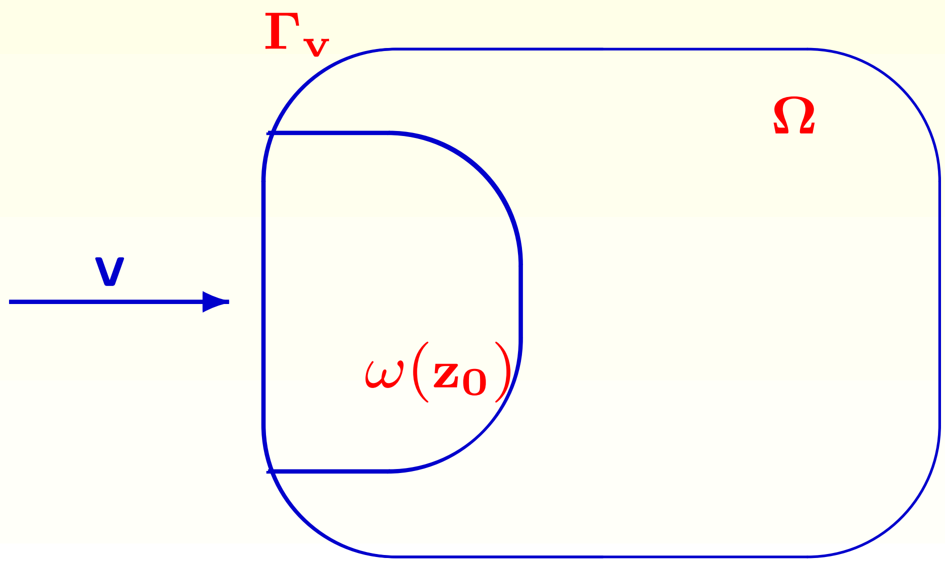
TRANSPORT SYSTEMS

Consider the transport equation

$$\left|\begin{array}{ll} \dfrac{\partial z}{\partial t} + v \cdot \nabla z = z\varphi(z,v) & (x \in \Omega, t > 0) \\ z(t)\,|_{\Gamma_v} = 0 & (t > 0) \\ z(0) = z_0 & \end{array}\right.$$

where $\Gamma_v = \{x \in \partial\Omega \mid v \cdot \nu(x) < 0\}$ **the absorbing zone and** $\nu(\cdot)$ **stands for the outward normal vector to** $\partial\Omega$**.**

The map to be spread : $\omega(z) = \{x \in \Omega | z(x) = 0\}$



AN OPTIMAL CONTROL APPROACH

Consider the linear system

$$\left| \begin{array}{l} \dot{z} + Az = Bv \\ z(0) = z_0 \in \mathrm{dom}(A) \subset Z \subset L^p(\Omega) \end{array} \right.$$

Let $\omega(z) = \{x \in \Omega | z(x) = 0\}$ **Then a spreading control may be obtained by solving the optimal control problem**

$$\left| \begin{array}{l} \min J_0(v, \sigma) \\ v \in \mathcal{V}, \sigma \in \Sigma \end{array} \right.$$

where $\Sigma \doteq \{\sigma = (\sigma_t)_t \subset \Omega | (\sigma_t)_t \textbf{increases}\}$ **and**

$$J_0(v, \sigma) \doteq \int_0^T \int_{\sigma_t} z^2(x, t, v) dx dt + \alpha \int_0^T |v|^2 dt + \beta F(\sigma)$$

FEEDBACK ANALYSIS BY SET-VALUED APPROACH

For each $z \in \mathcal{S}$ let $\mathcal{T}_\omega(z)$ be the set of y such that

$$(T) \left| \begin{array}{l} \forall\ \delta > 0, \exists\ 0 < h < \delta \text{ and } \|p\| \leq \delta \ \text{ such that} \\ S(h)z + h(y+p) \in \mathcal{S} \text{ and} \\ \omega(S(h)z + h(y+p)) \supset \omega(z); \end{array} \right.$$

and let

$$\mathcal{F}_\omega(z) \doteq \{v \in V \mid \varphi(z,v) \in \mathcal{T}_\omega(z)\}$$

Theorem : Assume :

- **the semigroup $S(\cdot)$ is compact**
- $\Sigma_\omega \doteq \{(y,z) \in \mathcal{S}^2 \mid \omega(y) \supset \omega(z)\}$ **is closed.**
- $\varphi(\cdot, \varsigma(\cdot))$ **is demicontinuous on $\mathcal{S}$.**

Then

ς is a FSC law with respect to ω iff ς is a selection of the map $\mathcal{F}_\omega$

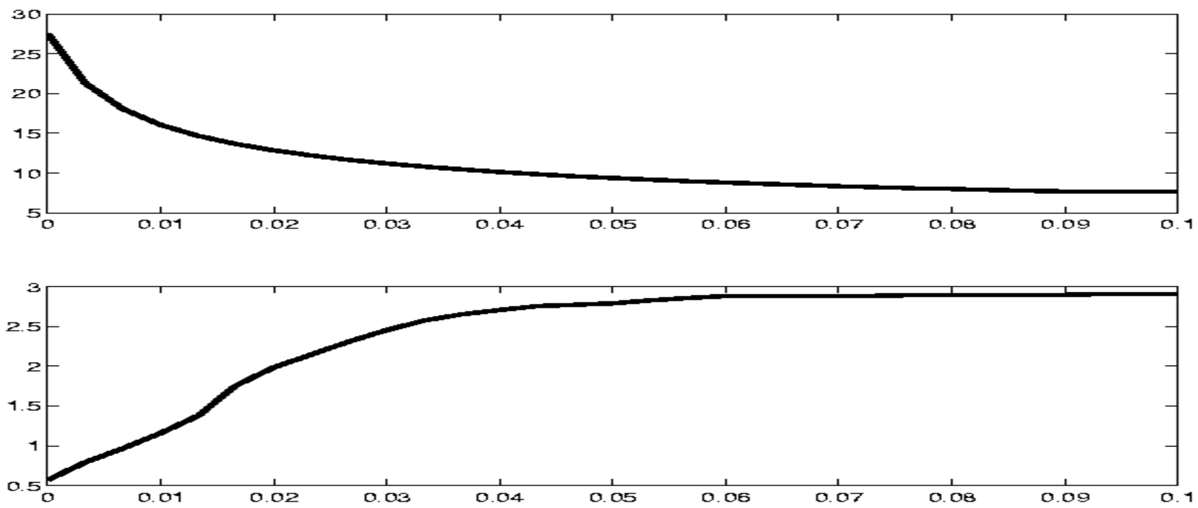
AN ALGORITHM

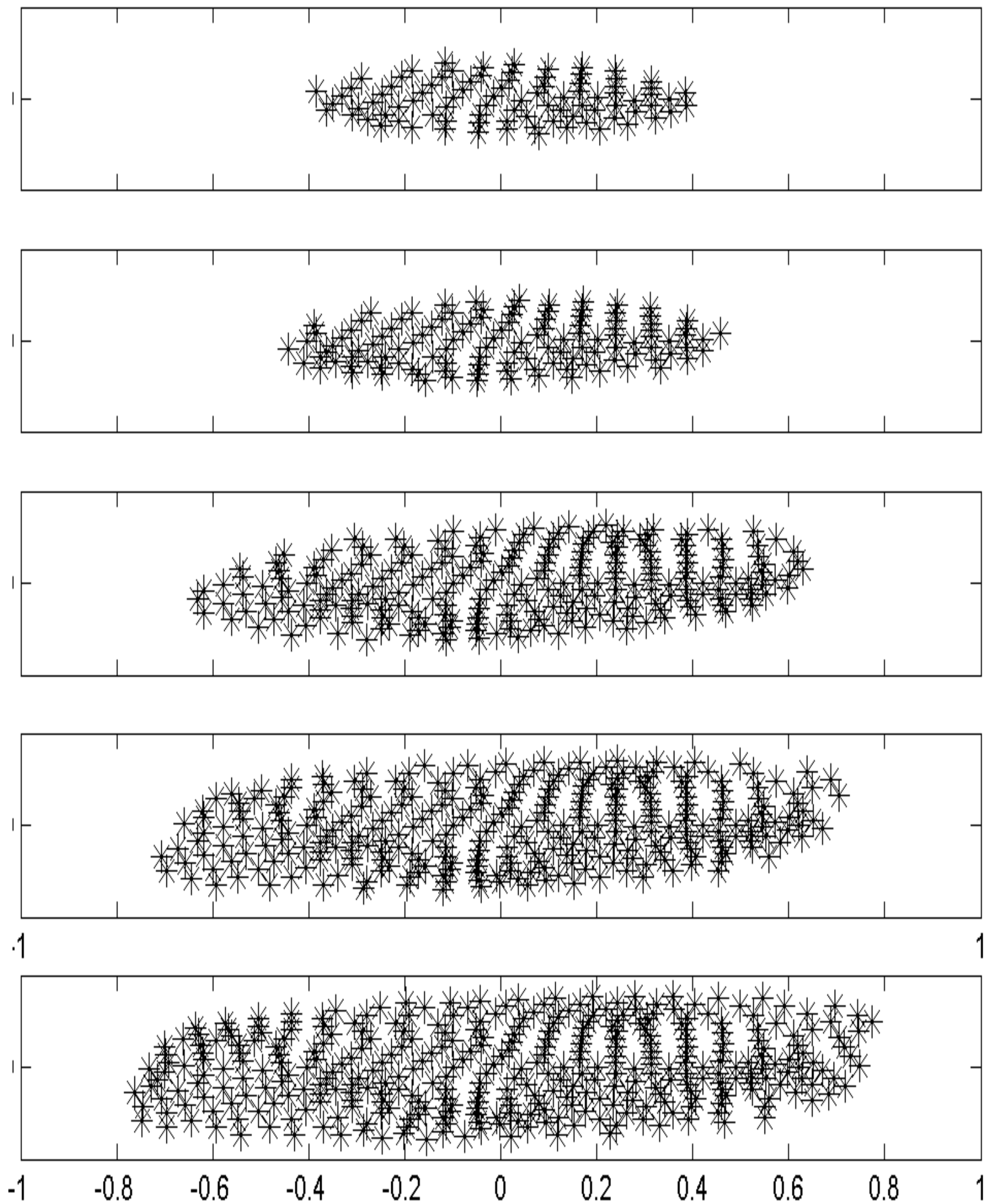
Given $h > 0, N \in \mathbb{N}$ such that $h = t_1/N$.

1. Initialize $r = z_0$ and $\sigma = \omega(z_0)$.
2. Iterate for $k = 1$ to N.
   - Find v such that $\omega(S(h)r + h\varphi(r, v)) \supset \sigma$.
   - Let $v_k = v$ and $z_k = S(h)r + h\varphi(r, v)$.
   - Put $r = z_k$ and $\sigma = \omega(z_k)$.
3. At each time $t_k = kh$:
   - The approximated state is z_k,
   - the spreading control is v_k,
   - the generated spread is $(\omega(z_k))_k$.

AN EXAMPLE

$$\left| \begin{array}{l} \frac{\partial u}{\partial t} - \Delta u = (4 + xy)v, \quad \Omega =]-1,1[\times]-1,1[\\ u = 0 \textbf{ on } \partial\Omega, \\ u_0(x,y) = \cos(\pi x) + \sin(\pi y), \omega(u) = \{u \geq 1.2\} \end{array} \right.$$





-1
1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

TAKING IN ACCOUNT THE SPREAD SPEED

Let μ be a measure on Ω. The speed functional is given by,

$$\theta(y,z) \doteq \liminf_{h\downarrow 0, \|p\|\to 0} \frac{\mu(\omega(S(h)z + h(y+p)) \setminus \omega(z))}{h}$$

for each $z \in \mathcal{S}$ and $y \in \mathcal{T}_\omega(z)$.

Immediate Properties :

- θ is well-defined and has values ranging in $[0,\infty]$.
- Assume $\tau_\omega \doteq \mu \circ \omega$ to be locally Lipschitz on $\mathcal{S}$, then,

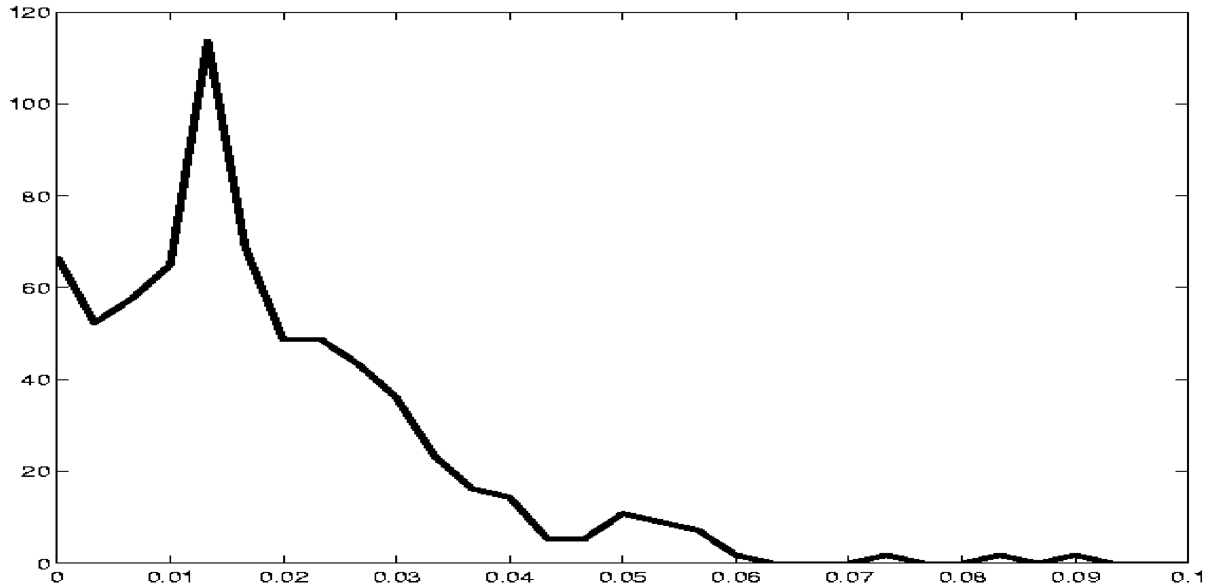
$$\mathrm{speed}(t,\bar{v}) = \theta(\varphi(\bar{z}(t), \bar{v}(t)), \bar{z}(t)) \quad (t \in [0, t_1[),$$

  where $\bar{z} = z(\cdot, \bar{v})$ and $\bar{v}$ be a spreading control.
- Suppose that $\mathcal{S}$ and τ_ω are convex, then, we have

$$\theta(y,z) = d\tau_\omega(z)(y - Az)$$

  for each $y \in \mathcal{T}_\omega(z)$ and $z \in \mathcal{S} \cap \mathrm{dom}(A)$.
  $d\tau_\omega$ the directional derivative of τ_ω.

The speed functional



FSC LAW WITH SPEED CONSTRAINTS

$\mathbf{P}_\nu$ | Find a FSC law $v = \varsigma_\nu(z)$ such that : $\rho(z, v) \geq \nu(z)$ for each $z \in \mathcal{S}$.

where

- $\nu(\cdot)$ a desired state dependent speed to exceed
- $\rho(z, v) = \theta(\varphi(z, v), v)$ the speed

Let

$$\mathcal{T}_\omega^\nu(z) \doteq \{y \in \mathcal{T}_\omega(z) \mid \theta(y, z) \geq \nu(z)\}$$

and

$$\mathcal{F}_\omega^\nu(z) \doteq \{v \in V \mid \varphi(z, v) \in \mathcal{T}_\omega^\nu(z)\}$$

Then selections of the map $\mathcal{F}_\omega^\nu$ may be solutions of Problem $\mathbf{P}_\nu$

Assume : Σ_ω is closed and the map ω^{-1} has convex values

Then Lemma :

- The map $\mathcal{T}_\omega$ has closed values.
- the map $\mathcal{T}_\omega$ has convex values.

A PRELIMINARY RESULT

Assume

- $\mathcal{T}_\omega$ is lsc.
- τ_ω is Gâteaux differentiable.
- ν is such that $\forall z \in \mathcal{S}, \exists y \in \mathcal{T}_\omega(z)$ such that $\theta(y, z) > \nu(z)$
- ν is upper semicontinuous

Then, the map $\mathcal{T}_\omega^\nu$ is lsc.

Proof outline :

- $\mathcal{T}_\omega^\nu$ is lsc $\Longleftrightarrow \kappa : z \in \mathcal{S} \to d(y_0, \mathcal{T}_\omega^\nu(z))^2 = \min\limits_{\substack{y \in \mathcal{T}_\omega(z) \\ \nu(z) - \theta(y,z) \leq 0}} \|y_0 - y\|^2$

  is upper semicontinuous for each $y_0 \in Y$.
- For $y_0 \in Y$ and $z \in \mathcal{S}$, we use,

$$\kappa(z) = \sup_{\lambda \geq 0} \inf_{y \in \mathcal{T}_\omega(z)} \{||y_0 - y||^2 + \lambda(\nu(z) - \theta(y, z))\}$$

- show that

$$\limsup_{n \to \infty} \kappa(z_n) \leq \kappa(z) \text{ if } z_n \to z$$

EXISTENCE FOR A SOLUTION AND ALGORITHM

Theorem : Assume $\forall z \in \mathcal{S}, y \in \mathcal{T}_\omega(z), \exists v \in V \mid \varphi(z,v) = y$.

Then, there exists a FSC law which solves Problem $\mathbf{P}_\nu$.

Proof due to Michael Selection Theorem.

AN ALGORITHM

Given $h > 0, N \in \mathbb{N}$ **such that** $h = t_1/N$.

1 **Initialize** $r = z_0$ **and** $\sigma = \omega(z_0)$.

2 **Iterate for** $k = 1$ **to** N.

- **Find** v **such that :**
  $\omega(S(h)r + h\varphi(r,v)) \supset \sigma$ **and** $\rho(z_k, v) \geq \nu(z_k)$
- **Let** $v_k = v$ **and** $z_k = S(h)r + h\varphi(r,v)$.
- **Put** $r = z_k$ **and** $\sigma = \omega(z_k)$.

3 **At each time** $t_k = kh$ **:**

- **The approximated state is** z_k,
- **the spreading control with speed constraint is** v_k,
- **the generated spread is** $(\omega(z_k))_k$.