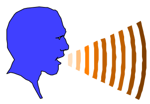
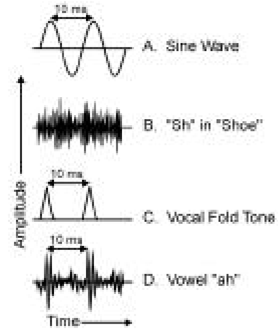
CHAPTER 5

COMPLEX WAVES



Preamble

- The sine wave is the fundamental component of all other sound waves
- All waves that are not sinusoidal are <u>complex waves</u>



FOURIER'S THEOREM

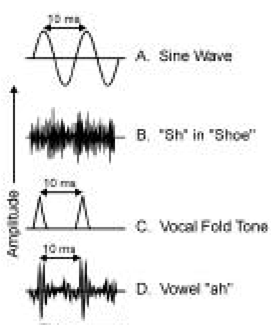
- A <u>complex wave</u> is any sound wave that is not sinusoidal
- Complex waves consist of a series of <u>simple sinusoids</u> that can differ in <u>amplitude</u>, <u>frequency</u>, & <u>phase</u>
- This is called a <u>Fourier series</u>
- A Fourier series can be derived by a process that is called <u>Fourier analysis</u>

Fourier Analysis

- Any complex wave can be decomposed to determine the <u>amplitudes</u>, <u>frequencies</u>, and <u>phases</u> of the sinusoidal components
- <u>All</u> sound waves can be classified by reference to:
  1. Is periodicity present?
  2. How complex is the wave?

PERIODIC WAVES

- A wave that repeats itself over time
- Also called a <u>periodic time function</u>
- A <u>complex periodic wave</u> is periodic, but not sinusoidal
  - Panels A, C, and D are periodic waves, though panel A is not <u>complex</u> periodic



Components of a Complex Periodic Wave

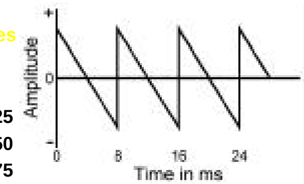
- Sinusoidal components cannot be selected at random--They must satisfy an <u>harmonic relation</u>
- With an harmonic relation, each sinusoid in the series must be an <u>integer multiple</u> of the lowest in the series
  - e.g., lowest = 100 Hz: Components are?
    - ☑ 100, 200, 300, 400, 500, etc.
  - e.g., lowest = 215 Hz: Components are?
    - ☑ 215, 430, 645, 860, etc.

Harmonic Series

- If harmonic relation is present, the series of components is called an <u>harmonic series</u>
- Each of the components is called an <u>harmonic</u>
  - 1st (and f_0), 2nd, 3rd, 4th, 5th, etc.

Harmonic Series

- T = 8 ms; f_0 = 125 Hz
- What are the frequencies of the first five harmonics?
  - 1st (f_0) = 125 × 1 = 125
  - 2nd = 125 × 2 = 250
  - 3rd = 125 × 3 = 375
  - 4th = 125 × 4 = 500
  - 5th = 125 × 5 = 625



Harmonics, Partials, and Overtones

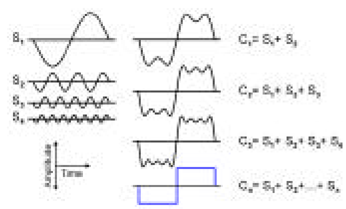
- If all components are <u>exact</u> integer multiples of f_0, harmonic # = partial #
- 2nd harmonic = 1st overtone

Table 5-1. Fundamental frequency, harmonics, partials, and overtones in a complex periodic sound wave

Frequency	Harmonic	Partial	Overtone
125 (f_0)	1	1	
250	2	2	1
375	3	3	2
500	4	4	3
625	5	5	4
750	6	6	5

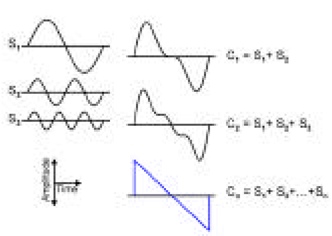
Summation of Sine Waves

- Progressive summation of 4 components with identical starting phases
- Each component is an <u>odd</u> integer multiple of f_0
- Compare S_1, C_1, C_2, C_3, & C_∞
- A <u>square wave</u> at ∞



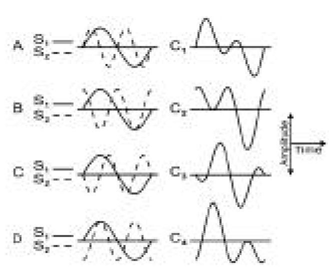
Summation of Sine Waves

- Progressive summation of 3 components with identical starting phases
- The components are <u>odd & even</u> integer multiples of f_0
- Compare S_1, C_1, C_2, & C_∞
- A <u>sawtooth wave</u> at ∞



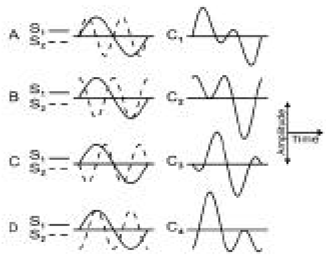
Summation of Sine Waves

- Summation of two components, S_1 and S_2
  - Starting phase of S_1 is held constant
  - Starting phase of S_2 varies



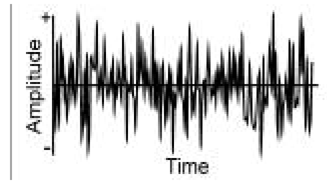
Summation of Sine Waves

- Note how shape of C changes with changes in starting phase of S_2
  - Panel A: 0^o
  - Panel B: 90^o
  - Panel C: 180^o
  - Panel D: 270^o



APERIODIC WAVES

- A wave that lacks periodicity
- Vibratory motion is <u>random</u>, & it sometimes is called a <u>random time function</u>
- Not necessarily “noise”

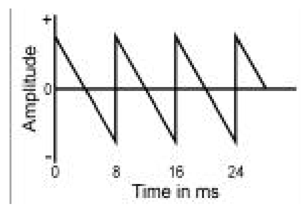


WAVEFORM AND SPECTRUM

- 1. Waveform
  - Plot of changes in some variable as a function of time
  - e.g., displacement, velocity, acceleration, pressure, etc. as a function of time

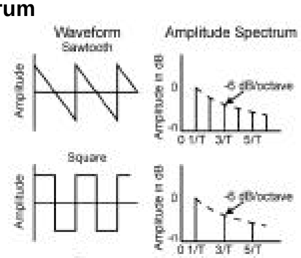
Waveform and Spectrum

- Observe the <u>fundamental period</u>: T = 8 ms
- Can calculate f_0 (125 Hz), but cannot easily see all frequency components, or their amplitudes or starting phases



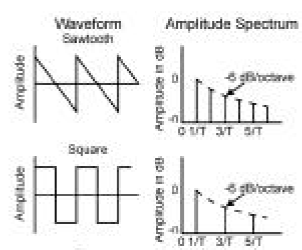
Waveform and Spectrum

- 2. Amplitude spectrum
  - A graphic alternative to the waveform
  - Also called the amplitude spectrum in the frequency domain



Waveform and Spectrum

- Shows amplitude as a function of frequency
- The spectral envelope is given by connecting the peaks of the vertical lines: in this case the slope of the envelope is - 6 dB / octave

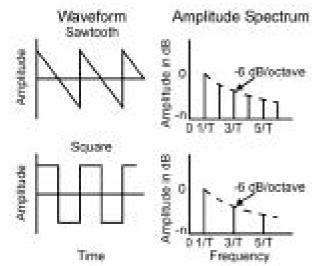


The Octave

- A doubling of frequency
- A frequency ratio of 2:1 or 1:2
  - 200:100
  - 500:250
  - 4000:2000
  - 100:200
  - 250:500
  - 2000:4000

Line Spectra

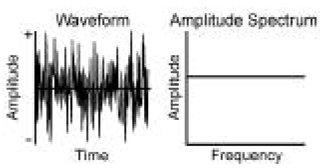
- In these cases, the amplitude spectrum is a <u>line spectrum</u>
  - Energy <u>only</u> at frequencies identified by vertical lines
  - Height of vertical line reflects amplitude



Continuous Spectra

white.wav

- Here the amplitude spectrum is a <u>continuous spectrum</u>
  - Energy present at all frequencies between certain frequency limits
  - What is slope of envelope?
    - ☑ 0 dB / octave
- A slope of 0 dB / octave is not a requirement for continuous spectra

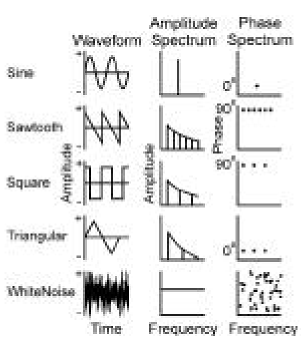


Waveform and Spectrum

- 3. Phase Spectrum
  - The phase spectrum in the frequency domain defines the <u>starting phase</u> as a function of frequency
  - The combination of the amplitude spectrum & the phase spectrum defines the waveform completely in the frequency domain

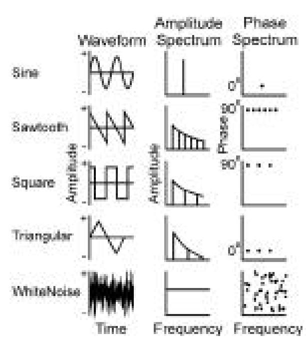
EXAMPLES OF COMPLEX WAVES

- Waveforms, amplitude spectra, & phase spectra of four complex waves
- Compare each with the reference sine wave at the top



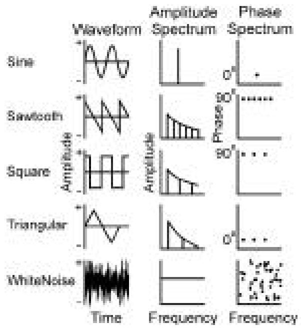
EXAMPLES OF COMPLEX WAVES

- Can you describe amplitude & phase spectra of the sine wave?
  - ☑ Amplitude spectrum: A line spectrum with energy at a single frequency
  - ☑ Phase spectrum: Starting phase is 0º (in this example)



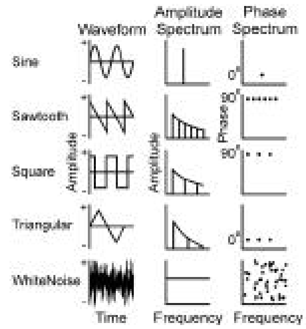
1. A Sawtooth Wave

- A complex periodic wave with energy at <u>odd</u> and <u>even</u> harmonics that has a <u>spectral envelope slope</u> of - 6 dB / octave
- Amplitudes decrease as the inverse of the harmonic #



Sawtooth Wave

- dB = 20 $\log_{10}(1/h_i)$, where h_i is the harmonic #
- dB = - 20 $\log_{10} h_i$
- Why?
  - ☑ Log Law 4



Sawtooth Wave

- Voltage of f_0 = 2v
- Voltage of other harmonics: $1/h_i \times 2$
  - 2nd harmonic: $1/2 \times 2$ = 1 v
  - 3rd harmonic = ?
    - ☑ $1/3 \times 2$ = .67 v
  - 4th harmonic = ?
    - ☑ $1/4 \times 2$ = .5 v

Table 5-2. Amplitudes (in voltage) of sinusoidal components of a **sawtooth** wave in which the amplitude of the fundamental frequency is 2 V

Harmonic Number	rms voltage ($1/h_i \times 2$)	-20 $\log_{10} h_i$
1 (f_0)	1/1 × 2 = 2	0
2	1/2 × 2 = 1	-6
3	1/3 × 2 = .67	-9.5
4	1/4 × 2 = .50	-12
5	1/5 × 2 = .40	-14
6	1/6 × 2 = .33	-15.6
7	1/7 × 2 = .29	-16.9
8	1/8 × 2 = .25	-18.1
9	1/9 × 2 = .22	-19.1

Sawtooth Wave

- What if voltage of f_0 is 4v?
- What are voltages of 2nd, 3rd, and 4th harmonics?
  - ☑ 2nd = $1/2 \times 4$ = 2v
  - ☑ 3rd = $1/3 \times 4$ = 1.33v
  - ☑ 4th = $1/4 \times 4$ = 1.0v

Table 5-2. Amplitudes (in voltage) of sinusoidal components of a **sawtooth** wave in which the amplitude of the fundamental frequency is 2 V

Harmonic Number	rms voltage ($1/h_i \times 2$)	-20 $\log_{10} h_i$
1 (f_0)	1/1 × 2 = 2	0
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8	1/8 × 2 = .25	-18.1
9	1/9 × 2 = .22	-19.1

Sawtooth Wave

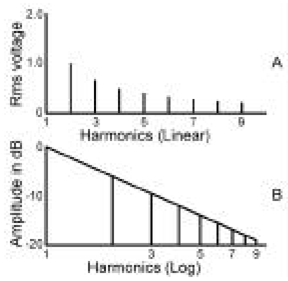
- <u>Absolute</u> voltage of any harmonic depends on voltage of f_0!
- <u>Relative level</u>, in dB, is independent of voltage of f_0!
  - 2nd = - 20 $\log_{10} 2$ = - 6 dB
  - 3rd = - 20 $\log_{10} 3$ = - 9.5 dB
  - 4th = ?
    - ☑ - 20 $\log_{10} 4$ = - 12 dB

Sawtooth Wave

- Comparison among harmonics
  - 2nd re: 1st (f_0) : - 6 dB
  - 4th re: 2nd : - 6 dB
  - 6th re: 3rd : - 6 dB
- The spectral envelope slope is - 6 dB / octave

Sawtooth Wave

- What does amplitude spectrum of a sawtooth wave "look like?"
  - Depends on choice of linear vs. log scales



Ch5-31

Summary of Sawtooth Wave

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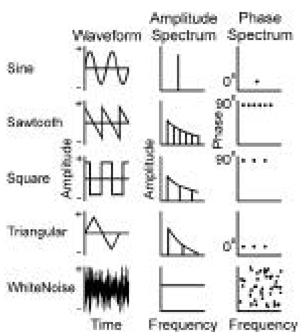
- A complex periodic wave
- Energy at odd & even integer multiples of f_0
- Spectral envelope slope of - 6 dB / octave
- $dB = 20 \log_{10} 1/h_i = -20 \log_{10} h_i$

Ch5-32

2. Square Wave

- A complex periodic wave with energy only at odd integer multiples of f_0 that has a spectral envelope slope of - 6 dB / octave



Ch5-33

Square Wave

- Amplitudes decrease as the inverse of the harmonic #
- $dB = 20 \log_{10} (1/h_i) = -20 \log_{10} h_i$

Table 5-3. Amplitudes (in voltage) of sinusoidal components of a **square** wave in which the amplitude of the fundamental frequency is 2 V

Harmonic Number	rms voltage ($1/h_i \times 2$)	$-20 \log_{10} h_i$
1 (f_0)	1/1 × 2 = 2	0
3	1/3 × 2 = .67	-9.5
5	1/5 × 2 = .40	-14
7	1/7 × 2 = .29	-16.9
9	1/9 × 2 = .22	-19.1

Ch5-34

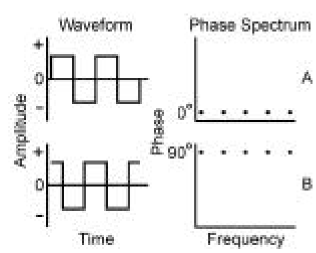
Summary of Square Wave

- Complex periodic wave
- Energy only at odd integer multiples of f_0
- Spectral envelope slope of - 6 dB / octave
- $dB = 20 \log_{10} 1/h_i = -20 \log_{10} h_i$

Ch5-35

Summary of Square Wave

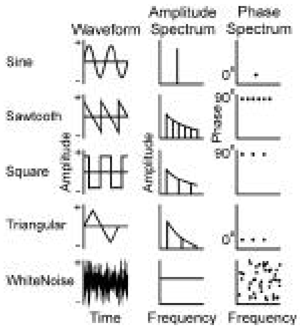
- What about the phase spectrum?
- Confusion among textbooks
  - Some will show the starting phase to be 0º; others as 90º
  - Each is correct, but all harmonics must have the same starting phase



Ch5-36

3. Triangular Wave

- A complex periodic wave with energy only at odd harmonics
- What distinguishes the triangular wave from the square wave?
  - ☑ Slope of envelope is steeper for triangular wave



Triangular Wave

- Amplitudes decrease as the reciprocal of the square of the harmonic #
- dB = $20 \log_{10} 1/h_i^2$
  $= -20 \log_{10} h_i^2$
  $= -40 \log_{10} h_i$
- Why - 40?
  - ☑ Log Law 3

Table 5-4. Amplitudes (in voltage) of sinusoidal components of a **triangular** wave in which the amplitude of the fundamental frequency is 2 V

Harmonic Number	rms voltage ($1/h_i^2 \times 2$)	$-40 \log_{10} h_i$
1 (f_0)	$1/1^2 \times 2 = 2$	0
3	$1/3^2 \times 2 =$ .22	-19.1
5	$1/5^2 \times 2 =$ .08	-28
7	$1/7^2 \times 2 =$ .04	-33.8
9	$1/9^2 \times 2 =$ .025	-38.2

Triangular Wave

- We saw earlier that the 5th harmonic of a square or sawtooth wave was -14 dB
- But, the 5th harmonic of triangular wave is -28 dB ($20 \log_{10} 1/5^2$)
- The spectral envelope slope is -12 dB / octave

Table 5-4. Amplitudes (in voltage) of sinusoidal components of a **triangular** wave in which the amplitude of the fundamental frequency is 2 V

Harmonic Number	rms voltage ($1/h_i^2 \times 2$)	$-40 \log_{10} h_i$
1 (f_0)	$1/1^2 \times 2 = 2$	0
3	$1/3^2 \times 2 =$ .22	-19.1
5	$1/5^2 \times 2 =$ .08	-28
7	$1/7^2 \times 2 =$ .04	-33.8
9	$1/9^2 \times 2 =$ .025	-38.2

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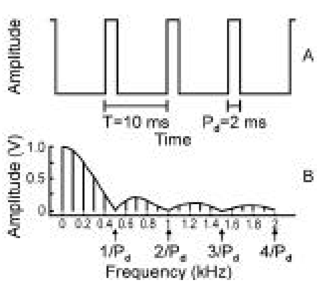
Summary of Triangular Wave

- Complex periodic wave
- Energy only at odd integer multiples of f_0
- Spectral envelope slope of -12 dB / octave
- dB = $20 \log_{10} 1/h_i^2 = -40 \log_{10} h_i$

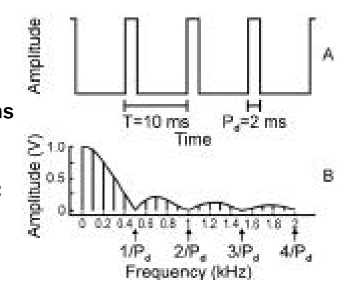
4. Pulse Train

- A repetitious series of rectangularly shaped pulses
- Each pulse has some width or duration (P_d)
- Is it periodic or aperiodic?
  - ☑ A line spectrum; it must be periodic



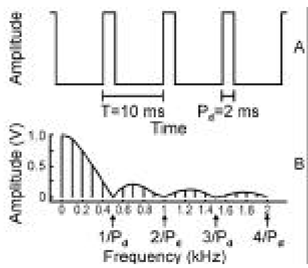
Pulse Train

- In this example, P_d = 2 ms
- The period (T) of the pulse is 10 ms
- 1/T defines the pulse repetition frequency (PRF):
  - PRF = ?
  - ☑ PRF = 100 Hz



Pulse Train

- A complex periodic wave with <u>harmonics</u> at odd and even integer multiples of the pulse repetition frequency: 100, 200, 300, etc.
- Amplitude spectrum shows <u>lobes</u> and <u>valleys</u> <u>(nulls)</u>
- <u>Nulls</u> occur at integer multiples of reciprocal of P_d

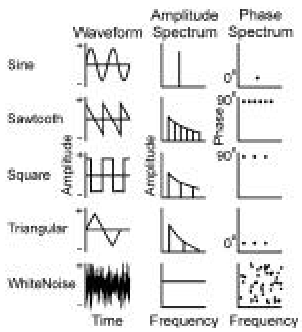


Pulse Train

- Thus, nulls occur at $1/P_d$, $2/P_d$, $3/P_d$, etc.
  - 500 Hz, 1000 Hz, 1500 Hz
- Starting phases?
  - Below 1st null: 0°
  - Between 1st and 2nd null: 180°
  - Between 2nd and 3rd null: 0°
  - and so forth

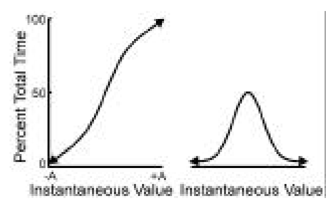
5. White, or Gaussian, Noise

- An aperiodic waveform with equal energy in every frequency band 1 Hz wide: from
  - f - 0.5 Hz to f + 0.5 Hz
- Why is this noise called "white noise?"
  - Analogous to white light -- equal energy in all light wavelengths



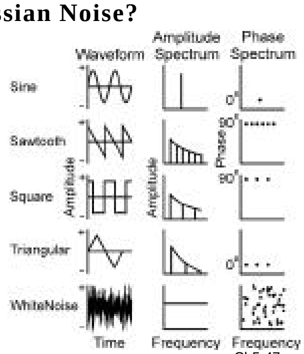
Why is White Noise Called Gaussian Noise?

- A random time function can be described by a <u>cumulative probability distribution</u> (left)
- A plot of the changing slope of a cumulative probability distribution is called a <u>probability density function</u> (right)



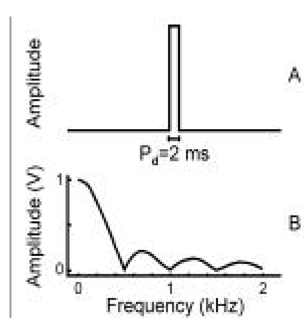
Why is White Noise Called Gaussian Noise?

- For white noise, the probability density function is a <u>normal curve</u>, or Gaussian distribution
- Spectral envelope slope of 0 dB/octave
- Starting phases in random array



6. A Single Pulse

- P_d = 2 ms
- Is the waveform periodic?
  - ☑ It cannot be!
  - ☑ It is <u>aperiodic</u>, & the amplitude spectrum must, therefore, be <u>continuous</u>



Pulse Train (Revisited)

● T (ms)	● PRF (Hz)
♦ 10	♦ 100
♦ 20	♦ 50
♦ 40	♦ 25
♦ 80	♦ 12.5
♦ 160	♦ 6.25
♦ 320	♦ 3.125
♦ ∞	♦ 0

- At T = ∞, PRF = 0 and the spacing between harmonics = 0
- The result is a continuous spectrum
- Nulls still occur at $1/P_d$, $2/P_d$, $3/P_d$, etc.

MEASURES OF SOUND PRESSURE FOR COMPLEX WAVES

- Different equations required for different signals
- True rms meter vs. average-responding meter
- Only the true rms meter will correctly read the rms voltage of signals other than sinusoids

Table 5-5. Measures of sound pressure for sine, square, and random waveforms. **A** refers to the peak or **maximum amplitude** as defined in Chapter 2

Metrics	Types of Waveforms		
	Sine	Square	Random
rms	$\frac{A}{\sqrt{2}}$	A	~ 0.3 A
mean square	$\frac{A^2}{2}$	A^2	~ 0.1 A
FW_{avg}	$\frac{2A}{\pi}$	A	~ .25 A
peak	A	A	A

SIGNAL-TO-NOISE RATIO IN dB (dB S/N)

- It is the ratio of signal level to noise level
- dB S / N = 10 $\log_{10}(I_S/I_N)$
- If S = 70 dB and N = 66 dB
  - ♦ dB S / N = 4 dB
- Why?
  - ☑ dB S / N = 10 $\log_{10}(10^{-5} / 4 \times 10^{-6})$ = +4 dB
  - ☑ dB S / N = 70 - 66 = +4 dB
  - ☑ (Log Law 2)