Análisis Vectorial – Coordenadas

Coordenadas Cartesianas

Posición, *velocidad* y *aceleración*:
$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$,
$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k} = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j} + \dfrac{dz}{dt}\mathbf{k}$,
$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k} = \dfrac{dv_x}{dt}\mathbf{i} + \dfrac{dv_y}{dt}\mathbf{j} + \dfrac{dv_z}{dt}\mathbf{k}$.

El **gradiente** de un campo escalar $U(x, y, z)$ es: $\vec{\nabla} U = \dfrac{\partial U}{\partial x}\mathbf{i} + \dfrac{\partial U}{\partial y}\mathbf{j} + \dfrac{\partial U}{\partial z}\mathbf{k}$.

La **diferencial total** es $dU = \dfrac{\partial U}{\partial x}dx + \dfrac{\partial U}{\partial y}dy + \dfrac{\partial U}{\partial z}dz$.

La **divergencia** y el **rotor** de un campo vectorial $\vec{F}\,(x, y, z) = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k}$ son:

$$\vec{\nabla}\bullet\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z},$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

El **laplaciano** es: $\nabla^2 U = \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + \dfrac{\partial^2 U}{\partial z^2}$.

Coordenadas Cilíndricas

Posición, *velocidad* y *aceleración*:
$\mathbf{r} = r\,\mathbf{e_r}$
$\mathbf{v} = v_r\,\mathbf{e_r} + v_\theta\,\mathbf{e_\theta} = \dfrac{dr}{dt}\mathbf{e_r} + r\dfrac{d\theta}{dt}\mathbf{e_\theta}$
$\mathbf{a} = a_r\,\mathbf{e_r} + a_\theta\,\mathbf{e_\theta}$ con:

$$a_r = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2 r}{dt^2} - r\omega^2,$$
$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = r\gamma + 2\frac{dr}{dt}\omega$$

El **gradiente** de un campo escalar $U(r,\theta, z)$ es: $\vec{\nabla} U = \dfrac{\partial U}{\partial r}\mathbf{e_r} + \dfrac{1}{r}\dfrac{\partial U}{\partial \theta}\mathbf{e_\theta} + \dfrac{\partial U}{\partial z}\mathbf{e_z}$.

La **divergencia** y el **rotor** de un campo vectorial $\vec{F}\ (r,\theta, z) = F_r\, \mathbf{e_r} + F_\theta\, \mathbf{e_\theta} + F_z\, \mathbf{e_z}$ son:

$$\vec{\nabla}\bullet\vec{F} = \frac{1}{r}\cdot\left[\frac{\partial}{\partial r}\left(r\,F_r\right)+\frac{\partial F_\theta}{\partial \theta}+\frac{\partial}{\partial z}\left(r\,F_z\right)\right],$$

$$\vec{\nabla}\times\vec{F} = \frac{1}{r}\cdot\begin{vmatrix} \mathbf{e_r} & \mathbf{e_\theta} & \mathbf{e_z} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}.$$

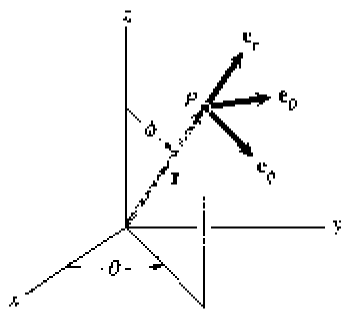
El **laplaciano** es: $\nabla^2 U = \dfrac{\partial^2 U}{\partial r^2}+\dfrac{1}{r}\cdot\dfrac{\partial U}{\partial r}+\dfrac{1}{r^2}\cdot\dfrac{\partial^2 U}{\partial \theta^2}+\dfrac{\partial^2 U}{\partial z^2}$.

Las *expresiones en coordenadas cartesianas* de los **versores cilíndricos** son:

$\mathbf{e_r} = \cos\theta\ \mathbf{i} + \operatorname{sen}\theta\ \mathbf{j} + z\ \mathbf{k}$ $\qquad$ $\mathbf{e}_\theta = -\operatorname{sen}\theta\ \mathbf{i} + \cos\theta\ \mathbf{j}$ $\qquad$ $\mathbf{e_z} = z\ \mathbf{k}$

<u>Coordenadas Esféricas</u>

Considerando r el vector radial, φ la inclinación medida desde el eje z hacia abajo y θ el ángulo entre el eje x y la proyección del vector r en el plano xy:



Posición, *velocidad* y *aceleración*:

$\mathbf{r} = r\ \mathbf{e_r}$ $\qquad$ $\mathbf{v} = \dfrac{dr}{dt}\mathbf{e_r} + r\dfrac{d\varphi}{dt}\mathbf{e_\varphi} + r\dfrac{d\theta}{dt}\operatorname{sen}\varphi\ \mathbf{e_\theta}$

$$\mathbf{a} = \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\varphi}{dt}\right)^2 - r\left(\frac{d\theta}{dt}\right)^2 \operatorname{sen}^2\varphi\right]\mathbf{e_r}$$
$$+\left[r\frac{d^2\varphi}{dt^2} + 2\frac{dr}{dt}\frac{d\varphi}{dt} - r\left(\frac{d\theta}{dt}\right)^2 \operatorname{sen}\varphi\cos\varphi\right]\mathbf{e_\varphi}$$
$$+\left[r\frac{d^2\theta}{dt^2}\operatorname{sen}\varphi + 2\frac{dr}{dt}\frac{d\theta}{dt}\operatorname{sen}\varphi + 2r\frac{d\varphi}{dt}\frac{d\theta}{dt}\cos\varphi\right]\mathbf{e_\theta}.$$

El **gradiente** de un campo escalar $U(r,\ \varphi,\ \theta)$ es: $\vec{\nabla}U = \dfrac{\partial U}{\partial r}\mathbf{e_r} + \dfrac{1}{r\operatorname{sen}\varphi}\dfrac{\partial U}{\partial \theta}\mathbf{e_\theta} + \dfrac{1}{r}\dfrac{\partial U}{\partial \varphi}\mathbf{e_\varphi}$.

La **divergencia** y el **rotor** de un campo vectorial $\vec{F}\ (r,\ \varphi,\ \theta) = F_r\, \mathbf{e_r} + F_\theta\, \mathbf{e_\theta} + F_\varphi\, \mathbf{e_\varphi}$ son:

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\operatorname{sen}\varphi}\cdot\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r\operatorname{sen}\varphi}\cdot\frac{\partial}{\partial \varphi}\left(F_\varphi \operatorname{sen}\varphi\right),$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{r^2 \operatorname{sen} \varphi} \cdot \begin{vmatrix} \mathbf{e_r} & r\mathbf{e}_\varphi & r \operatorname{sen} \varphi \, \mathbf{e}_\theta \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial \theta} \\ F_r & rF_\varphi & r \operatorname{sen} \varphi \, F_\theta \end{vmatrix}$$

El **laplaciano** es: $\nabla^2 U = \dfrac{\partial^2 U}{\partial r^2} + \dfrac{2}{r} \cdot \dfrac{\partial U}{\partial r} + \dfrac{1}{r^2 \operatorname{sen}^2 \varphi} \cdot \dfrac{\partial^2 U}{\partial \theta^2} + \dfrac{1}{r^2} \cdot \dfrac{\partial^2 U}{\partial \varphi^2} + \dfrac{\operatorname{cotg} \varphi}{r^2} \cdot \dfrac{\partial U}{\partial \varphi}$.

Las *expresiones en coordenadas cartesianas* de los **versores esféricos** son:

$\mathbf{e_r} = \operatorname{sen} \varphi \cos \theta \, \mathbf{i} + \operatorname{sen} \varphi \operatorname{sen} \theta \, \mathbf{j} + \cos \varphi \, \mathbf{k}$ $\quad$ $\mathbf{e}_\varphi = \cos \varphi \cos \theta \, \mathbf{i} + \cos \varphi \operatorname{sen} \theta \, \mathbf{j} - \operatorname{sen} \varphi \, \mathbf{k}$

$\mathbf{e}_\theta = -\operatorname{sen} \theta \, \mathbf{i} + \cos \theta \, \mathbf{j}$ $\quad$ Obs.: $\mathbf{e_r} \times \mathbf{e}_\varphi = \mathbf{e}_\theta$

Coordenadas Intrínsecas

La *velocidad* y la *aceleración* son:

$\mathbf{v} = v \, \mathbf{e_t} = \dfrac{ds}{dt} \mathbf{e_t}$ $\quad$ $\mathbf{a} = a_t \, \mathbf{e_t} + a_n \, \mathbf{e_n}$ donde:

$a_t = \dfrac{dv}{dt}$, $a_n = v \dfrac{d\theta}{dt} = \dfrac{v^2}{\rho}$.

Fuentes:

- "***Dinámica***" – *Bedford-Fowler* – Apéndice
- "***Advanced Engineering Mathematics***" – *Edwin Kreyszig* – John Wiley & Sons (6º ed. – pág. 498)