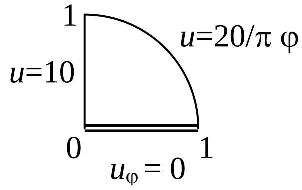
Coloquio
Análisis Matemático IIIC (61.13)

1) Calcular $u(r,0)$ con:

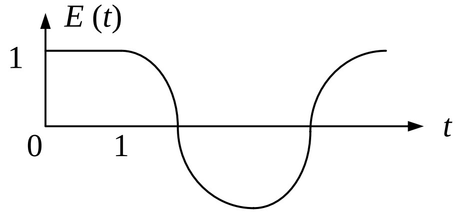
$$\nabla^2 u = 0\,,\ u(r,\pi/2) = 10\,,\ u(1,\varphi) = \frac{20}{\pi}\varphi\,,\ \frac{\partial u}{\partial \varphi}(r,0) = 0\,.$$

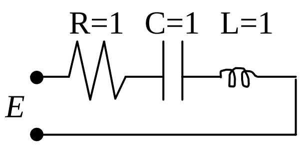


2)
   a) Hallar la solución general de la ecuación diferencial en $V(0)$: $3xy'' + 2y' + y = 0$
   b) Hallar la solución de la ecuación diferencial holomorfa en 0.

3) Hallar $f(t)$, $-\infty < t < \infty$, si $F(w) = \dfrac{1}{(w-2i)(w+3i)}$.

4) Hallar $I(t)$ y graficarla:





$I(0) = 0$

$$RI(t) + L\frac{dI}{dt} + \frac{1}{C}\int_0^t I(t)\,dt = E(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 < t < 1 \\ \cos(t-1) & t > 1 \end{cases}$$

Respuestas

1) (En polares): $\nabla^2 v(r,\varphi) = v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\varphi\varphi} = 0$

   Propongo $v(r,\varphi) = R(r).\Phi(\varphi)$ para resolver las condiciones homogeneas:

   $v(r,\pi/2) = 0$, $\quad \dfrac{\partial v}{\partial \varphi}(r,0) = 0$

$$\Rightarrow\ R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0 \ \to\ R''\Phi + \frac{1}{r}R'\Phi = -\frac{1}{r^2}R\Phi'' \to$$

$$\frac{R''\Phi}{R\Phi}r^2 + \frac{1}{r}\frac{R'\Phi}{R\Phi}r^2 = -\frac{1}{r^2}\frac{R\Phi''}{R\Phi}r^2 \to \frac{R''}{R}r^2 + \frac{R'}{R}r = -\frac{\Phi''}{\Phi} = \lambda\ \ (cte)$$

$$\Rightarrow \begin{cases} r^2R'' + rR' - \lambda R = 0 \\ \Phi'' + \lambda\Phi = 0 \end{cases}$$

   Propongo $\Phi(\varphi) = Ae^{\alpha\varphi}$

$$\Rightarrow Ae^{\alpha\varphi}(\alpha^2 + \lambda) = 0 \to \alpha = \pm\sqrt{\lambda}\,i \qquad \Rightarrow\ \Phi(\varphi) = A\cos\left(\sqrt{\lambda}\,\varphi\right) + B\,\mathrm{sen}\left(\sqrt{\lambda}\,\varphi\right)$$

$v(r,\pi/2) = R(r)\,\Phi(\pi/2) = 0 \Rightarrow \Phi(\pi/2) = 0 \qquad \dfrac{\partial v}{\partial \varphi}(r,0) = R(r)\,\Phi'(0) = 0 \Rightarrow \Phi'(0) = 0$

$\Rightarrow$B = 0 $\quad \Rightarrow \sqrt{\lambda}\,\pi/2 = \pi/2 + n\,\pi \Rightarrow \lambda = (2n+1)^2$

$\Rightarrow \Phi_n(\varphi) = A_n \cos\big((2n+1)\varphi\big)$

Para la ec. de Euler, solución: $R(r) = \sum_{n=0}^{\infty} C_n r^n$

$$r^2\sum_{n=0}^{\infty} n(n-1)C_n r^{n-2} + r\sum_{n=0}^{\infty} nC_n r^{n-1} - \lambda\sum_{n=0}^{\infty} C_n r^n = 0$$

Coef. r^n: $n(n\text{-}1)C_n + nC_n - \lambda\, C_n = 0 \to C_n\,(n^2 - \lambda) = 0 \Rightarrow C_n = 0 \quad \forall n \neq \sqrt{\lambda}$

$\Rightarrow R_n(r) = D_n\, r^{2n+1} + E_n\, r^{-2n-1}$ Para que sea holomorfa, $E_n = 0 \Rightarrow R_n(r) = D_n\, r^{2n+1}$

$$\therefore\ v(r,\varphi) = \sum_{n=0}^{\infty} R_n(r).\Phi_n(\varphi) = \sum_{n=0}^{\infty} D_n r^{2n+1} A_n \cos\big((2n+1)\varphi\big)$$

$v(r,\varphi) = \sum_{n=0}^{\infty} C_n \cos\big((2n+1)\varphi\big)\, r^{2n+1}$, que cumple las condiciones homogeneas.

Ahora, propongo $u(r,\varphi) = v(r,\varphi) + w(\varphi)$ con la $v(r,\varphi)$ anterior.

Entonces: $\nabla^2 u(r,\varphi) = \nabla^2 v(r,\varphi) + w''(\varphi) = 0 + w''(\varphi) = 0 \qquad \Rightarrow w''(\varphi) = 0$

$\Rightarrow w(\varphi) = \alpha\,\varphi + \beta$

$u(r,\pi/2) = v(r,\pi/2) + w(\pi/2) = 0 + w(\pi/2) = 10 \qquad \Rightarrow w(\pi/2) = 10$

$\dfrac{\partial u}{\partial \varphi}(r,0) = \dfrac{\partial v}{\partial \varphi}(r,0) + w'(0) = 0 + w'(0) = 0 \qquad \Rightarrow w'(0) = 0$

$\therefore w(\varphi) = 10$

Aplico la última condición:

$u(1,\varphi) = v(1,\varphi) + w(\varphi) = \dfrac{20}{\pi}\varphi \qquad \Rightarrow \qquad v(1,\varphi) = u(1,\varphi) - w(\varphi) = \dfrac{20}{\pi}\varphi - 10$

$$\Rightarrow\ v(1,\varphi) = \sum_{n=0}^{\infty} C_n \cos\big((2n+1)\varphi\big) = \frac{20}{\pi}\varphi - 10$$

Por el teorema de Sturm-Liouville, $\Phi_n(\varphi) = C_n \cos((2n+1)\varphi)$ son un familia ortogonal de funciones en $(0,\pi/2)$ con núcleo 1.

$\Rightarrow$ Puedo usar la serie de Fourier generalizada: $C_n = \dfrac{\big(f(\varphi),\cos\big((2n+1)\varphi\big)\big)}{\big\|\cos\big((2n+1)\varphi\big)\big\|^2}$,

siendo $f(\varphi) = \dfrac{20}{\pi}\varphi - 10$.

$$\Rightarrow C_n = \frac{\int_0^{\pi/2}\left(\frac{20}{\pi}\varphi - 10\right)\cos\big((2n+1)\varphi\big)\,d\varphi}{\int_0^{\pi/2}\cos^2\big((2n+1)\varphi\big)\,d\varphi} = \frac{\dfrac{-20}{\pi(2n+1)^2}}{\dfrac{\pi}{4}}$$

$$\boxed{u(r,\varphi) = \sum_{n=0}^{\infty}\left(\frac{-80}{(2n+1)^2}\cos\big((2n+1)\varphi\big)\, r^{2n+1}\right) + 10}$$

2) a) Propongo $y(x)=\sum_{n=0}^{\infty} C_n x^{n+r}$

$$\Rightarrow\ 3x\sum_{n=0}^{\infty}(n+r)(n-1+r)C_n x^{n+r-2}+2\sum_{n=0}^{\infty}(n+r)C_n x^{n+r-1}+\sum_{n=0}^{\infty}C_n x^{n+r}=0$$

Coef. x^{n+r-1}: $3(n+r)(n-1+r)C_n + 2(n+r)C_n + C_{n-1} = 0$

$C_n\,[3(n+r)(n-1+r) + 2(n+r)] + C_{n-1} = 0$

Con $n=0$: $C_0\,[3r(r-1) + 2r] + C_{-1} = 0$ Por hip., $C_{-1} = 0$.

Entonces, $3r^2 - 3r + 2r = 0$. $r_1 = 1/3$ $r_2 = 0$

Con $r=1/3$: $C_n\,[3(n+1/3)(n-2/3) + 2(n+1/3)] + C_{n-1} = 0$

$C_n\,[3n^2 + n - 2/3 + 2n + 2/3] + C_{n-1} = 0$

$C_n\,[3n^2 + 2n] + C_{n-1} = 0$

$$(n\neq 0, n\neq \tfrac{2}{3}),\quad C_n=\frac{-C_{n-1}}{n(3n+1)}$$

C_0 = arbitrario $\quad C_1=\frac{-C_0}{1.4}\quad C_2=\frac{-C_1}{2.7}=\frac{C_0}{1.2\,.\,4.7}\quad C_3=\frac{-C_2}{3.10}=\frac{-C_0}{1.2.3\,.\,4.7.10}$

$$\Rightarrow (n>1):\ C_n=\frac{(-1)^n}{n!}\cdot\frac{C_0}{4.7.10.\ldots.(3n+1)}$$

$$\Rightarrow\ y_1(x)=\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n!}\cdot\frac{C_0}{4.7.10.\ldots.(3n+1)}\cdot x^{n+\frac{1}{3}}\right)+C_0$$

Con $r = 0$: $C_n\,[3n(n-1) + 2n] + C_{n-1} = 0$

$C_n\,[3n^2 - 3n + 2n] + C_{n-1} = 0$

$C_n\,[3n^2 - n] + C_{n-1} = 0$

$$\left(n\neq \tfrac{1}{3}, n\neq 0\right)\quad C_n=\frac{-C_{n-1}}{n(3n-1)}$$

C_0 = arbitrario $\quad C_1=\frac{-C_0}{1.2}\quad C_2=\frac{-C_1}{2.5}=\frac{C_0}{1.2\,.\,2.5}\quad C_3=\frac{-C_2}{3.8}=\frac{-C_0}{1.2.3\,.\,2.5.8}$

$$\Rightarrow (n>1):\ C_n=\frac{(-1)^n}{n!}\cdot\frac{C_0}{2.5.8.\ldots.(3n-1)}$$

$$\Rightarrow\ y_2(x)=\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n!}\cdot\frac{C_0}{2.5.8.\ldots.(3n-1)}\cdot x^{n}\right)+C_0$$

$\therefore$ Sol. gral. en $V(0)$:

$$y(x)=\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n!}\cdot\frac{A}{2.5.8.\ldots.(3n-1)}\cdot x^{n}\right)+A+\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n!}\cdot\frac{B}{2.5.8.\ldots.(3n-1)}\cdot x^{n+\frac{1}{3}}\right)+B$$

$r = 1/3$ hace que la solución NO sea holomorfa pues tiene un punto de ramificación la función (compleja). $\Rightarrow$ Esa solución la descarto ($y_1(x)$).

$$\therefore\ \ y(x)=\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n!}\cdot\frac{A}{2.5.8.\ldots.(3n-1)}\cdot x^n\right)+A$$

3) $f(t)=\mathcal{F}^{-1}\left[F(w)\right]=\frac{1}{2\pi}VP\int_{-\infty}^{\infty}F(w).e^{iwt}dw$

Para $t>0$, camino por semiplano complejo superior:

$$\oint\frac{e^{izt}}{(z-2i)(z+3i)}dz=\int_{-R}^{R}\frac{e^{iwt}}{(w-2i)(w+3i)}dw+\int_{\Gamma_R}\frac{e^{izt}}{(z-2i)(z+3i)}dz=2\pi i\,R_{z=2i}\left[\frac{e^{izt}}{(z-2i)(z+3i)}\right]$$

Por el T. de Jordan, $\lim\limits_{R\to\infty}\int_{\Gamma_R}\frac{e^{izt}}{(z-2i)(z+3i)}dz=0$ pues $\lim\limits_{z\to\infty}\frac{1}{(z-2i)(z+3i)}=0$ y $t>0$.

$$R_{z=2i}\left[\frac{e^{izt}}{(z-2i)(z+3i)}\right]=\lim_{z\to 2i}\frac{(z-2i)e^{izt}}{(z-2i)(z+3i)}=\frac{e^{-2t}}{5i}\qquad\Rightarrow\qquad f(t)=\tfrac{1}{5}e^{-2t}\ \text{si}\ t>0$$

Para $t<0$, camino por semiplano complejo inferior (T. de Residuos cambia de signo):

$$\oint\frac{e^{izt}}{(z-2i)(z+3i)}dz=\int_{-R}^{R}\frac{e^{iwt}}{(w-2i)(w+3i)}dw+\int_{\Gamma_R}\frac{e^{izt}}{(z-2i)(z+3i)}dz=-2\pi i\,R_{z=3i}\left[\frac{e^{izt}}{(z-2i)(z+3i)}\right]$$

Por el T. de Jordan, $\lim\limits_{R\to\infty}\int_{\Gamma_R}\frac{e^{izt}}{(z-2i)(z+3i)}dz=0$ pues $\lim\limits_{z\to\infty}\frac{1}{(z-2i)(z+3i)}=0$ y $t<0$.

$$R_{z=-3i}\left[\frac{e^{izt}}{(z-2i)(z+3i)}\right]=\lim_{z\to -3i}\frac{(z+3i)e^{izt}}{(z-2i)(z+3i)}=\frac{e^{3t}}{-5i}\qquad\Rightarrow\qquad f(t)=\tfrac{1}{5}e^{3t}\ \text{si}\ t<0$$

$$\therefore\ \boxed{f(t)=\begin{cases}\tfrac{1}{5}e^{3t} & t<0\\ \tfrac{1}{5}e^{-2t} & t>0\end{cases}}$$

4) Aplicando Laplace:
$RQ(s)$ + $L[sQ(s)$ – $I(0)]$ + $1/C$ $Q(s)/s$ = $W(s)$ siendo $Q(s)$ la transformada de $I(t)$ y $W(s)$ la transformada-Laplace de $E(t)$.
Reemplazando los valores de R, L y C:

$$Q+sQ+Q/s=W\qquad\Rightarrow\qquad Q(s)=\frac{sW}{s^2+s+1}$$

$$W(s)=\int_0^{\infty}E(t)e^{-st}dt=\int_0^1 e^{-st}dt+\int_1^{\infty}\cos(t-1)e^{-st}dt=\frac{1}{s}-\frac{e^{-s}}{s}+A(s)$$

Para calcular $A(s)$: $\int_1^{\infty}e^{i(t-1)}e^{-st}dt=e^{-i}\int_1^{\infty}e^{(i-s)t}dt=e^{-i}\left.\frac{e^{(i-s)t}}{i-s}\right|_1^{\infty}=e^{-i}\left(0-\frac{e^{i-s}}{i-s}\right)=$

$$=e^{-i}e^{i}\frac{e^{-s}}{s-i}=\frac{e^{-s}}{s-i}=\frac{e^{-s}}{s^2+1}\left(s+i\right)=\frac{se^{-s}}{s^2+1}+i\frac{e^{-s}}{s^2+1}=A(s)+i\,B(s)$$

$A(s)$ corresponde a $\cos(t\text{-}1)$ y $B(s)$ a $\text{sen}(t\text{-}1)$.

$$\Rightarrow\ Q(s)=\frac{\left(1-e^{-s}\right)}{s^2+s+1}+\frac{s^2e^{-s}}{\left(s^2+s+1\right)\left(s^2+1\right)}$$

$s^2 + s + 1 = s^2 + s + ¼ + ¾ = (s + ½)^2 + ¾$

Fracciones simples: $\dfrac{\alpha s+\beta}{s^2+s+1}+\dfrac{\delta s+\gamma}{s^2+1}=\dfrac{\left(\alpha s+\beta\right)\left(s^2+1\right)+\left(s^2+s+1\right)\left(\delta s+\gamma\right)}{\left(s^2+s+1\right)\left(s^2+1\right)}$

$$=\frac{\alpha(s^3+s)+\beta(s^2+1)+\delta(s^3+s^2+s)+\gamma(s^2+s+1)}{\left(s^2+s+1\right)\left(s^2+1\right)}=$$

$$=\frac{(\alpha+\delta)s^3+(\beta+\delta+\gamma)s^2+(\alpha+\delta+\gamma)s+(\beta+\gamma)}{\left(s^2+s+1\right)\left(s^2+1\right)}\qquad\Rightarrow\begin{cases}\alpha+\delta=0\\ \beta+\delta+\gamma=0\\ \alpha+\delta+\gamma=1\\ \beta+\gamma=0\end{cases}\Rightarrow\begin{cases}\alpha=0\\ \beta=-1\\ \delta=0\\ \gamma=1\end{cases}$$

$$\Rightarrow Q(s)=\frac{\left(1-e^{-s}\right)}{s^2+s+1}+e^{-s}\left(\frac{-s}{s^2+s+1}+\frac{s}{s^2+1}\right)$$

$$Q(s)=\left(1-e^{-s}\right)\left(\frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}\right)+e^{-s}\left(-\frac{s+1/2}{(s+1/2)^2+3/4}+\frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}+\frac{s}{s^2+1}\right)$$

$$Q(s)=\frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}+e^{-s}\left(-\frac{s+1/2}{(s+1/2)^2+3/4}-\frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+3/4}+\frac{s}{s^2+1}\right)$$

$\therefore$

$$\boxed{I(t)=\frac{2}{\sqrt{3}}\text{sen}\left(\tfrac{\sqrt{3}}{2}t\right)e^{-\frac{1}{2}t}H(t)+\left(-\cos\left(\tfrac{\sqrt{3}}{2}(t-1)\right)e^{-\frac{1}{2}(t-1)}-\frac{1}{\sqrt{3}}\text{sen}\left(\tfrac{\sqrt{3}}{2}(t-1)\right)e^{-\frac{1}{2}(t-1)}+\cos(t-1)\right)H(t-1)}$$