Advanced Calculus
Sequences and series: summary

1 Sequences

A *sequence* $\{a_n\}$ is an infinite list of numbers $a_1, a_2, a_3 \ldots, a_n, \ldots$. More formally a sequence is a function $f : \mathbf{N} \to \mathbf{R}$; however we prefer to use the subscript notation a_n for the n-th term of the sequence rather than the functional notation $f(n)$.

1.1 Basic properties of sequences

A sequence $\{a_n\}$ is *bounded above* if there is a number M such that $a_n \leq M$ for every n; such an M is an *upper bound* for the sequence. Similarly $\{a_n\}$ is *bounded below* if there is a number m such that $m \leq a_n$ for every n; this m is an *lower bound* for the sequence. A sequence which is both bounded above and bounded below is called simply a *bounded* sequence.

A sequence $\{a_n\}$ is *increasing* if $a_n < a_{n+1}$ for each n, and is *nondecreasing* if $a_n \leq a_{n+1}$ for each n. Similarly $\{a_n\}$ is *decreasing* if $a_n > a_{n+1}$ for each n, and is *nonincreasing* if $a_n \geq a_{n+1}$ for each n. A sequence that is nondecreasing or nonincreasing is called a *monotone* sequence.

1.2 Convergence

The notion of convergence is the most important notion in the theory of sequences. Let $\{a_n\}$ be a sequence and A a real number. We say that $\{a_n\}$ *converges* to A if for all positive real numbers δ there is a number N such that $|a_n - A| < \delta$ for all n with $n \geq A$.

What this means is that if we take any open interval $I = (A - \delta, A + \delta)$ centred at A then all but finitely many of the terms of the sequence a_n lie in I.

If $\{a_n\}$ converges to A we write $a_n \to A$ ("a_n tends to A") or $\lim_{n\to\infty} a_n = A$ ("the limit of a_n as n tends to infinity is A").

A sequence which converges to some real number is called *convergent*. A sequence which is not convergent is called *divergent*.

As an example of a convergent sequence define $a_n = 1/n$. This sequence converges to 0. To see that the definition of convergence is satisfied, let $\delta > 0$ and $I = (-\delta, \delta)$. We need to show that $a_n \in I$ with only finitely many exceptions. To this end consider the real number $1/\delta$. Choose an integer N with $N > 1/\delta$. If $n \geq N$ then $n > 1/\delta$ and so $0 < 1/n < \delta$. Hence $a_n \in I$ for all $n \geq N$, that is $a_n \in I$ with only finitely many exceptions.

In practice we rarely check convergence by verifying the definition. Instead we use rules whereby the convergence of known sequences imply the convergence of other series.

Constant sequences: If $a_n = a$ for all n, then $a_n \to a$.

Sum/difference/product/quotient rules: Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences converging to A and B respectively. Then

$$(a_n + b_n) \to A + B, \qquad (a_n - b_n) \to A - B, \qquad (a_n b_n) \to AB,$$

and if $b_n \neq 0$ for all n and $B \neq 0$ then

$$a_n/b_n \to A/B.$$

Sandwich rule: Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are all sequences, and that $a_n \leq b_n \leq c_n$ for all n. If $a_n \to A$ and $c_n \to A$ (that is they converge to the **same** limit) then $b_n \to A$ too.

Bounded monotone sequences: Every bounded monotone sequence is convergent.

Some divergent sequence fail to converge in a fairly disciplined way. We say that a sequence $\{a_n\}$ "diverges to ∞" and we write $a_n \to \infty$ if for each real number K there is a number N such that $a_n > K$ whenever $n \geq N$. That is to say that all but finitely many of the terms of the sequence are bigger than K, no matter what K is. Similarly $\{a_n\}$ "diverges to $-\infty$" ($a_n \to -\infty$) if for each real number K there is a number N such that $a_n < K$ whenever $n \geq N$.

It is also useful to remember a few standard limits.

1. $a^n \to 0$ if $|a| < 1$, and $a^n \to \infty$ if $a > 1$,
2. $n^k \to 0$ if $k < 0$, and $n^k \to \infty$ if $k > 1$,
3. $\dfrac{\ln n}{n^k} \to 0$ if $k > 0$,
4. $\left(1 + \dfrac{a}{n}\right)^n \to e^a$ for any constant a,
5. $n^{1/n} \to 1$.

2 Series

A *series* is an "infinite" sum

$$\sum_{n=1}^{\infty} = a_1 + a_2 + \cdots + a_n + \cdots$$

of a sequence $\{a_n\}$ of terms. To make sense of this we introduce the *partial sums*

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

We say that the sequence $\sum_{n=1}^{\infty} a_n$ *converges* if the sequence $\{s_n\}$ of its partial sums converges and we write

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$$

if this is the case. Otherwise the series $\sum_{n=1}^{\infty} a_n$ *diverges.*

An important example of series is the geometric series given by $a_n = ar^{n-1}$. That is $a_1 = 1$ and $a_{n+1} = ra_n$ for each n. To consider its convergence we shall compute the partial sums s_n. We have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and so

$$rs_n = ar + ar^2 + ar^3 + \cdots + ar^n.$$

Subtracting gives

$$(1 - r)s_n = a - ar^n \qquad \text{or} \qquad s_n = \frac{a(1 - r^n)}{1 - r}$$

(at least if $r \neq 1$). If $|r| < 1$ then $r^n \to 0$ and by using the difference, product and quotient rules we get $a_n \to a/(1 - r)$. Hence if $|r| < 1$ the geometric series is convergent and

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r}.$$

2.1 Convergence tests for positive series

If a series $\sum_{n=1}^{\infty} a_n$ is convergent, then the sequence of its partial sums $\{s_n\}$ is convergent to a number A say. Also $s_{n-1} \to A$ and so $s_n - s_{n-1} \to A - A = 0$. Hence $a_n \to 0$. We get the first of the following convergence/divergence tests.

divergence test: If $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$; equivalently if $a_n \not\to 0$ then $\sum_{n=1}^{\infty}$ diverges.

comparison test: Suppose that $0 \leq a_n \leq b_n$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. Also if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

limit comparison test: Suppose that $a_n > 0$ and $b_n > 0$. If $(a_n/b_n) \to C$ where $0 < C < \infty$, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge or diverge.

If $(a_n/b_n) \to 0$ and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

If $(a_n/b_n) \to \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

integral test: Let f be a decreasing function defined on the interval $[1, \infty)$; the decreasing condition means that $f(x) > f(y)$ whenever $1 \leq x < y$. If $a_n = f(n)$ then the series $\sum_{k=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent.

ratio test: Let $a_n > 0$ for each n. Suppose that $(a_{n+1}/a_n) \to k$. If $k < 1$ the series $\sum_{n=1}^{\infty} a_n$ converges; if $k > 1$ or if $a_{n+1}/a_n \to \infty$ the series $\sum_{n=1}^{\infty} a_n$ diverges; if $k = 1$ the test gives no information.

An an example consider the integral test with $f(x) = 1/x^k$ for $k > 0$. Then f is a decreasing function. We need to evaluate the integral $I_k = \int_1^{\infty} dx/x^k$. For $k = 1$, $I_1 = [\ln x]_1^{\infty} = \infty$ is divergent. Hence the *harmonic series* $\sum_{n=1}^{\infty} 1/n$ diverges. If $k \neq 1$ then $I_k = [x^{1-k}/(1-k)]_1^{\infty}$. The behaviour of the integral depends on the sign of $1 - k$. If $1 - k < 0$, that is if $k > 1$ then x^{1-k} goes to 0 as x goes to infinity. Hence $I_k = 1/(k-1)$ is convergent. But if $1 - k > 0$ that is if $0 < k < 1$ then x^{1-k} goes to infinity as x goes to infinity and $I_k = \infty$ is divergent. Hence we have that

$$\sum_{n=1}^{\infty} \frac{1}{n^k} \quad \begin{cases} \text{is convergent if } k > 1, \\ \text{is divergent if } 0 < k \leq 1. \end{cases}$$

This result is worth remembering, as the convergence of many series can be tested by comparison to these series.

2.2 Series with mixed signs

If $\sum_{n=1}^{\infty} |a_n|$ is convergent then so is $\sum_{n=1}^{\infty} a_n$. In this case we call the series $\sum_{n=1}^{\infty} a_n$ *absolutely convergent.* Given a series $\sum_{n=1}^{\infty} a_n$ with positive and negative terms, always first test $\sum_{n=1}^{\infty} |a_n|$ for convergence. If so the original series is convergent.

However the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent despite the series $\sum_{n=1}^{\infty} 1/n$ being divergent. Such a series is *conditionally convergent.* Many series like this with signs alternating between plus and minus are convergent due to the *alternating series test* or *Leibniz's test*:
Suppose that $\{a_n\}$ is a sequence such that

1. $\{a_n\}$ is nondecreasing: $a_n \geq a_{n+1}$, and
2. $a_n \to 0$,

then

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

is convergent.

2.3 Power series

These are series of the form

$$\sum_{n=0}^{\infty} a_n x^n.$$

We ask for which values of x these series converge for. Let $b_n = a_n x^n$. Then

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x|.$$

If $|a_{n+1}/a_n| \to L$ where L is a positive limit, then $\sum a_n x^n$ converges when $L|x| < 1$ and diverges for $L|x| > 1$, that is the series converges when $-R < x < R$ and diverges for $|x| < R$ where $R = 1/L$. We call R the *radius of convergence* of $\sum a_n x^n$.

If $|a_{n+1}/a_n| \to 0$ then $|\frac{b_{n+1}}{b_n}| \to 0$ too and so the power series $\sum a_n x^n$ converges for all x.

If $|a_{n+1}/a_n| \to \infty$ then the power series diverges for all nonzero x (but converges for $x = 0$).

For all power series $\sum_{n=0}^{\infty} a_n x^n$, there is a radius of convergence R, which is either a real number $R \geq 0$ or $R = \infty$ such that either

- $R = \infty$ and $\sum_{n=0}^{\infty} a_n x^n$ converges for all x,
- $R = 0$ and $\sum_{n=0}^{\infty} a_n x^n$ converges only for $x = 0$,
- $0 < R < \infty$ and $\sum_{n=0}^{\infty} a_n x^n$ converges when $-R < x < R$ and diverges when $|x| > R$. (It may converge or diverge when $x = \pm R$.)

Power series can be differentiated and integrated within their radius of convergence. Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

has radius of convergence R. The power series

$$\sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$$

also has radius of convergence R, and equals $f'(x)$ whenever $|x| < R$. Also the power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} = \sum_{m=1}^{\infty} \frac{a_{m-1}}{m} x^m$$

has radius of convergence R and equals

$$\int_0^x f(t)\,dt,$$

the indefinite integral of f.

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