Advanced Calculus
Differential equations: summary

1 First order equations

1.1 Separable equations

These have the form

$$\frac{dy}{dx} = f(x)g(y).$$

We "separate the variables" and integrate

$$\begin{aligned} \frac{dy}{g(y)} &= f(x)\,dx, \\ \int \frac{dy}{g(y)} &= \int f(x)\,dx + C. \end{aligned}$$

1.2 Homogeneous equations

These have the form

$$\frac{dy}{dx} = F(x, y)$$

where $F(tx, ty) = F(x, y)$. We substitute $y = xv$ to get

$$v + x\frac{dv}{dx} = F(x, xv) = F(1, v)$$

or

$$\frac{dv}{dx} = \frac{F(1, v) - v}{x}.$$

This is now a separable equation for v. Separate the variables to solve.

1.3 Linear equations

These have the form

$$\frac{dy}{dx} + p(x)y = q(x). \qquad (*)$$

We introduce the integrating factor

$$\phi(x) = \exp\left(\int p(x)\,dx\right).$$

Then

$$\phi'(x) = p(x)\phi(x)$$

and so

$$\frac{d}{dx}(\phi(x)y) = \phi(x)\frac{dy}{dx} + \phi'(x)y = \phi(x)\left(\frac{dy}{dx} + p(x)y\right).$$

By $(*)$ then

$$\frac{d}{dx}(\phi(x)y) = \phi(x)q(x).$$

We integrate this to determine $\phi(x)y$ and so also y.

2 Linear equations with constant coefficients

These have the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = q(x)$$

where $a_0, \dots, a_n$ are constants.

If $q(x) = 0$ the equation is *homogeneous linear.*

2.1 The homogeneous case

First consider the second order case

$$a_2 y'' + a_1 y' + a_0 y = 0. \tag{$\dagger$}$$

If $y = e^{mx}$ then

$$a_2 y'' + a_1 y' + a_0 y = (a_2 m^2 + a_1 m + a_0) e^{mx}$$

so this is a solution provided that m is a root of the quadratic equation

$$a_2 m^2 + a_1 m + a_0 = 0$$

which is called the *auxiliary equation.* If the auxiliary equation has distinct real roots m_1 and m_2 then ($\dagger$) has general solution

$$y = A_1 e^{m_1 x} + a_2 e^{m_2 x}.$$

If the roots are the auxiliary equation are non-real: $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ then the above solution is valid, but it may be more conveniently re-expressed as

$$y = e^{\alpha x}(B_1 \cos \beta x + B_2 \sin \beta x).$$

If the auxiliary solution has a repeated root $m_1 = m_2 = m$ then the above is not the general solution, rather it is

$$y = (A + Bx) e^{mx}.$$

Now consider the n-th order case

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

The auxiliary equation is

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0.$$

If it has distinct real roots $m_1, \dots, m_n$ then the general solution is

$$y = A_1 e^{m_1 x} + \cdots + A_n e^{m_n x}.$$

If the auxiliary equation has a repeated real root m with multiplicity k then the contribution of m to the general solution is

$$(B_0 + B_1 x + \cdots + B_{k-1} x^{k-1}) e^{mx}.$$

If the auxiliary equation has a pair of non-repeated complex roots $\alpha \pm i\beta$ then their contribution of m to the general solution is

$$e^{\alpha x}(A \cos \beta x + B \sin \beta x).$$

However if each of these complex roots is repeated with multiplicity k then the contribution is

$$e^{\alpha x}[(A_0 + A_1 x + \cdots + A_{k-1} x^{k-1}) \cos \beta x + (B_0 + B_1 x + \cdots + B_{k-1} x^{k-1}) \sin \beta x].$$

2.2 The inhomogeneous case

Consider

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = q(x). \qquad (*)$$

Thee corresponding homogeneous equation is

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \qquad (\dagger)$$

The general solution of $(*)$ is $y = y_{CF} + y_{PI}$ where y_{CF}, the *complementary function*, is the general solution of $(\dagger)$ and y_{PI}, the *particular integral*, is any particular solution of $(\dagger)$.

To find a particular integral we use the method of undetermined coefficients (= systematic guesswork!). We write down a likely solution with indeterminates for coefficients, plug into $(*)$ and derive linear equations for these indeterminates.

If $q(x)$ is a polynomial of degree n we try $y = c_0 + c_1 x + \cdots + c_n x^n$.

If $q(x) = ae^{mx}$ and m is not a root of the auxiliary equation we try $y = ce^{mx}$. However, if m is a root of the auxiliary equation, with multiplicity k, we try $y = cx^k e^{mx}$.

If $q(x) = r(x)e^{mx}$ where $r(x)$ is a polynomial of degree n try $y = (c_0 + c_1 x + \cdots + c_n x^n)e^{mx}$ if m is not a root of the auxiliary equation. However, if m is a root of the auxiliary equation, with multiplicity k, we try $y = (c_0 + c_1 x + \cdots + c_n x^n)x^k e^{mx}$.

If $q(x) = ae^{\alpha x}\cos\beta x$ or $ae^{\alpha x}\sin\beta x$ try $y = e^{\alpha x}(b\cos\beta x + c\sin\beta x)$ if m is not a root of the auxiliary equation. The cases where m is a root of the auxiliary equation and the cases where $q(x) = r(x)e^{\alpha x}\cos\beta x$ or $r(x)e^{\alpha x}\sin\beta x$ with $r(x)$ a polynomial, I'll leave you to figure out.

3 Other types of equation

3.1 Euler's equation

This is

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_1 x y' + a_0 y = q(x)$$

where $a_0, \ldots, a_n$ are constants. This can be reduced to the case of constant coefficients by setting $x = e^t$. For convenience we only consider the second-order case

Consider

$$ax^2 y'' + bxy' + c = q(x). \qquad (*)$$

Set $x = e^t$ and write $\dot{y}$ for dy/dt and $\ddot{y}$ for d^2y/dt^2. Note that $dx/dt = e^t = x$ so that $dt/dx = 1/x$. Then

$$y' = \frac{dy}{dx} = \frac{dt}{dx}\frac{dx}{dt} = \frac{\dot{y}}{x}.$$

Hence

$$y'' = \frac{d}{dx}\frac{\dot{y}}{x} = -\frac{\dot{y}}{x^2} + \frac{1}{x}\frac{d\dot{y}}{dx} = -\frac{\dot{y}}{x^2} + \frac{1}{x}\frac{dt}{dx}\frac{d\dot{y}}{dt} = -\frac{\dot{y}}{x^2} + \frac{\ddot{y}}{x^2}.$$

Thus $xy' = \dot{y}$ and $x^2y'' = \ddot{y} - \dot{y}$. Now $(*)$ becomes

$$a\ddot{y} + (b-a)\dot{y} + cy = q(e^t).$$

This is now a constant coefficient equation and can be solved as such.

3.2 A particular solution is known

Consider the second-order homogeneous linear equation, with non-constant coefficients:

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{$*$}$$

If we know a particular solution $y(x) = u(x)$, substitute $y(x) = u(x)v(x)$ in $(*)$. After some manipulation $(*)$ becomes

$$A(x)v''(x) + B(x)v'(x) = 0$$

for some functions $A(x)$ and $B(x)$. This is now a first-order linear equation for $v'(x)$. It can be solved for $v'(x)$ and then we integrate to get $v(x)$. Finally we recall that $y(x) = u(x)v(x)$.

3.3 Simultaneous equations

Suppose we have a pair of constant coefficient equations for two dependent variables y and z:

$$\begin{aligned} y' &= ay + bz + p(x) \\ z' &= ry + sz + q(x). \end{aligned}$$

Write D for the operator d/dx. The equations become

$$\begin{aligned} (D-a)y - bz &= p(x) \\ -ry + (D-s)z &= q(x). \end{aligned}$$

We can manipulate these carefully to eliminate one unknown. We multiply the first by $(D-s)$ and the second by b to get

$$\begin{aligned} (D-s)(D-a)y - b(D-s)z &= (D-s)p(x) \\ -bry + b(D-s)z &= bq(x). \end{aligned}$$

Adding now gives a second-order linear equation with constant coefficients for y. This can be solved, and then z can be obtained by substituting in the original equations.

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