Inter American University of Puerto Rico
Bayamón Campus
School of Engineering
Department of Electrical Engineering

ELEN 3302 – Electric Circuits II

Problem Set 2 Solutions

Due Wednesday, September 1

Problem 1: Find the particular solution of $x(t)$ in

$$a\frac{d^2}{dt^2}x(t) + b\frac{d}{dt}x(t) + cx(t) = \cos\omega t$$

assuming as a solution:

(A) $x(t) = K\cos(\omega t + \phi)$, where K and ϕ are real.

Start by calculating the derivatives:

$$\begin{aligned} x(t) &= K\cos(\omega t + \phi) \\ \dot{x}(t) &= -K\omega\sin(\omega t + \phi) \\ \ddot{x}(t) &= -K\omega^2\cos(\omega t + \phi) \end{aligned}$$

Using the angle addition formulas:

$$\begin{aligned} x(t) &= K\cos\phi\cos\omega t - K\sin\phi\sin\omega t \\ \dot{x}(t) &= -K\omega\cos\phi\sin\omega t - K\omega\sin\phi\cos\omega t \\ \ddot{x}(t) &= K\omega^2\cos\phi\cos\omega t + K\omega^2\sin\phi\sin\omega t \end{aligned}$$

We introduce these expressions into the differential equation and equate sines and cosines. This yields two algebraic equations:

$$\begin{aligned} \text{cosine equation: } & cK\cos\phi - aK\omega^2\cos\phi - bK\omega\sin\phi = 1 \\ \text{sine equation: } & -c\sin\phi + a\omega^2\sin\phi - b\omega\cos\phi = 0 \end{aligned}$$

The sine equation yields:

$$\tan\phi = \frac{b\omega}{a\omega^2 - c}$$

Thus,

$$\sin\phi = \frac{b\omega}{\sqrt{(a\omega^2 - c)^2 + (b\omega)^2}}$$
$$\cos\phi = \frac{a\omega^2 - c}{\sqrt{(a\omega^2 - c)^2 + (b\omega)^2}}$$

Substituting these expressions into the cosine equation yields:

$$K = -\frac{1}{\sqrt{(a\omega^2 - c)^2 + (b\omega)^2}}$$

Thus, the particular solution is:

$$x(t) = -\frac{1}{\sqrt{(a\omega^2 - c)^2 + (b\omega)^2}} \cos(\omega t + tan^{-1}\frac{b\omega}{a\omega^2 - c})$$

(B) $x(t) = Ke^{j\omega t} + K^* e^{-j\omega t}$, where K and K^* are complex and K^* denotes the complex conjugate of K.

Again, start by calculating the derivatives:

$$x(t) = Ke^{j\omega t} + K^* e^{-j\omega t}$$
$$\dot{x}(t) = j\omega K e^{j\omega t} - j\omega K^* e^{-j\omega t}$$
$$\ddot{x}(t) = -\omega^2 K e^{j\omega t} - \omega^2 K^* e^{-j\omega t}$$

We introduce these expressions into the differential equation, express $\cos\omega t$ as $1/2e^{j\omega t} + 1/2e^{-j\omega t}$, and equate $e^{j\omega t}$ and $e^{-j\omega t}$ terms. This yields two complex algebraic equations:

$$e^{j\omega t} \text{ equation: } cK + bj\omega K - a\omega^2 K = 1/2$$
$$e^{-j\omega t} \text{ equation: } cK^* - bj\omega K^* - a\omega^2 K^* = 1/2$$

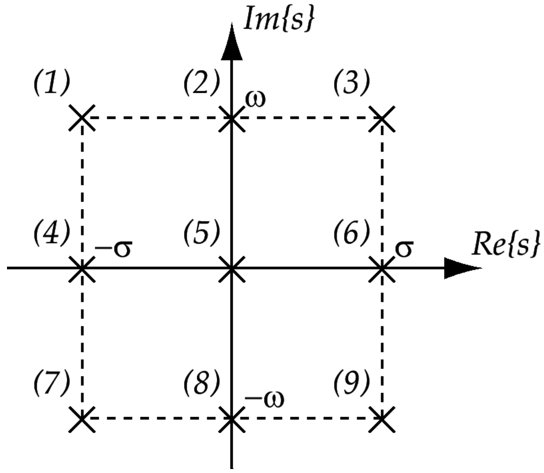
Note that one equation is the complex conjugate of the other, as expected. Solving for K and K^*:

$$K = \frac{1/2}{c - a\omega^2 + bj\omega} \qquad K^* = \frac{1/2}{c - a\omega^2 - bj\omega}$$

The particular solution is:

$$x(t) = \frac{1/2}{c - a\omega^2 + bj\omega} e^{j\omega t} + \frac{1/2}{c - a\omega^2 - bj\omega} e^{-j\omega t}$$

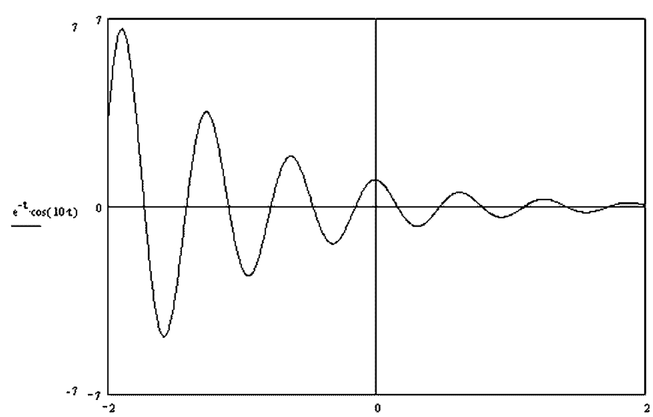
Problem 2: Provide an algebraic expression and a qualitative sketch of the real part of the complex time function for each of the complex frequencies marked with an $\times$ in the s-plane shown below:



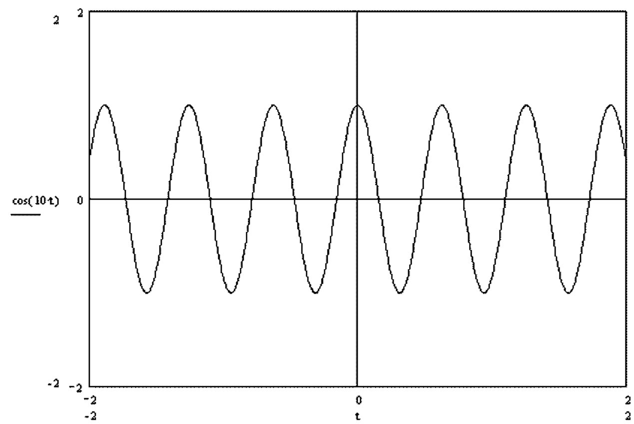
Note that each $s = \sigma + j\omega$ represents a complex time function of the form e^{st}. Since we are interested in the real part only, let's compute it and then substitute the different values of σ and ω:

$$
\begin{aligned}
Re\{e^{st}\} &= Re\{e^{(\sigma+j\omega)t}\} \\
&= Re\{e^{\sigma t}e^{j\omega t}\} \\
&= Re\{e^{\sigma t}(\cos\omega t + j\sin\omega t)\} \\
&= Re\{e^{\sigma t}\cos\omega t + je^{\sigma t}\sin\omega t\} \\
&= e^{\sigma t}\cos\omega t
\end{aligned}
$$

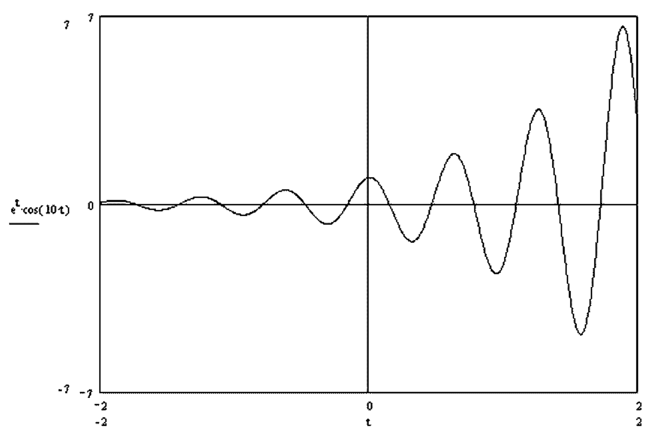
Therefore,
(1) $Re\{e^{st}\} = e^{-\sigma t}\cos\omega t$, exponentionally decaying oscillation



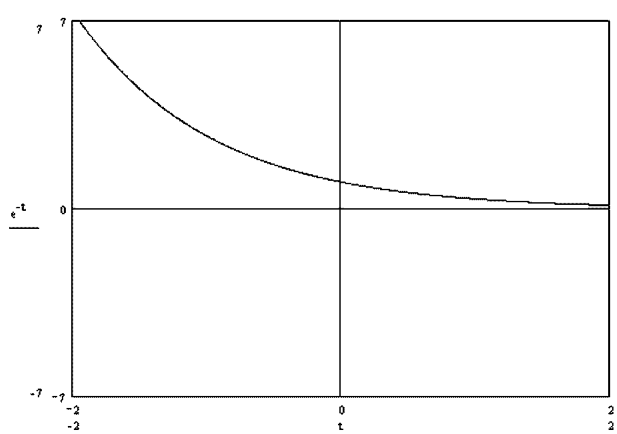
(2) $Re\{e^{st}\} = \cos \omega t$, constant oscillation



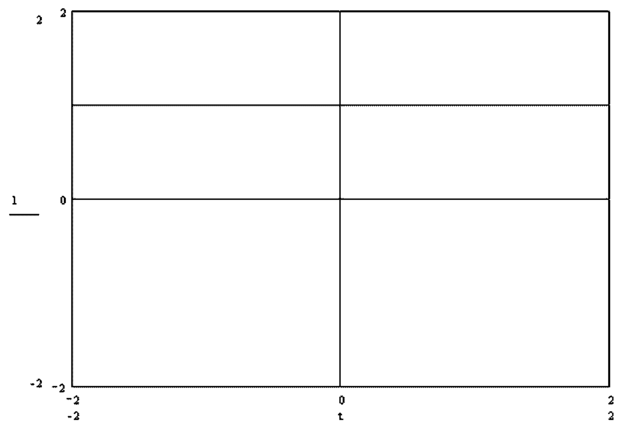
(3) $Re\{e^{st}\} = e^{\sigma t} \cos \omega t$, exponentionally increasing oscillation



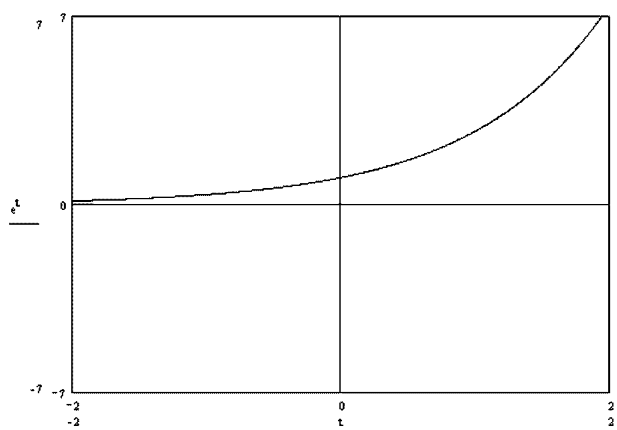
(4) $Re\{e^{st}\} = e^{-\sigma t}$, exponentional decay



(5) $Re\{e^{st}\} = 1$, constant

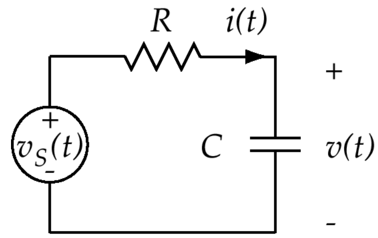


(6) $Re\{e^{st}\} = e^{\sigma t}$, exponentional increase



The frequency ω shows inside a cosine. Thus, the real part of a negative frequency will be exactly the same as the real part of the same positive frequency. Therefore, (7), (8) and (9) are identical to (1), (2) and (3), respectively.

Problem 3: Provide an algebraic expression for the sinusoidal steady-state of $i(t)$ and $v(t)$ if $v_S(t) = V_S \sin \omega t$.



Recall that the impedance of a resistor is R and a capacitor is $\frac{1}{Cj\omega}$. The relation between V and V_S is a voltage divider:

$$\frac{V}{V_S} = \frac{\frac{1}{Cj\omega}}{R + \frac{1}{Cj\omega}} = \frac{1}{RCj\omega + 1}$$

$$\left|\frac{V}{V_S}\right| = \frac{1}{\sqrt{1 + (RC\omega)^2}} \qquad \angle\frac{V}{V_S} = -\tan^{-1} RC\omega$$

$$v(t) = V_S \frac{1}{\sqrt{1 + (RC\omega)^2}} \sin(\omega t - \tan^{-1} RC\omega)$$

The current I can be found dividing V_S by the total impedance of the circuit:

$$\frac{I}{V_S} = \frac{1}{R + \frac{1}{Cj\omega}} = \frac{Cj\omega}{RCj\omega + 1}$$

$$\left|\frac{I}{V_S}\right| = \frac{C\omega}{\sqrt{1 + (RC\omega)^2}} \qquad \angle\frac{I}{V_S} = \frac{\pi}{2} - \tan^{-1} RC\omega$$

$$i(t) = V_S \frac{C\omega}{\sqrt{1 + (RC\omega)^2}} \sin(\omega t + \frac{\pi}{2} - \tan^{-1} RC\omega)$$