Inter American University of Puerto Rico
Bayamón Campus
School of Engineering
Department of Electrical Engineering

ELEN 3302 – Electric Circuits II

Problem Set 1 Solutions

Due Wednesday, August 25

Problem 1:

(A) Prove that $e^{jx} = \cos x + j\sin x$ by using the Taylor series expansion of e^x, $\sin x$ and $\cos x$.

Taylor expansions:
$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + ...$
$\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$
$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$

Evaluate e^{jx}:
$e^{jx} = \frac{(jx)^0}{0!} + \frac{(jx)^1}{1!} + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \frac{(jx)^5}{5!} + ...$
$e^{jx} = \frac{x^0}{0!} + j\frac{x^1}{1!} - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} + ...$
$e^{jx} = \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - ...\right) + j\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)$
$e^{jx} = \cos x + j\sin x$

(B) Using the relation proved in the previous part, derive the following relationships:

(i) $\cos x = \frac{1}{2}(e^{jx} + e^{-jx})$
$\cos x = \frac{\cos x + j\sin x + \cos(-x) + j\sin(-x)}{2}$
$\cos x = \frac{\cos x + j\sin x + \cos x - j\sin x}{2}$
$\cos x = \frac{2\cos x}{2}$
$\cos x = \cos x$

(ii) $\sin x = \frac{j}{2}(e^{-jx} - e^{jx})$ *(Hint: Show that $1/j = -j$.)*
$\sin x = \frac{j}{2}(\cos x - j\sin x - \cos x - j\sin x)$
$\sin x = \frac{j}{2}(-2j\sin x)$
$\sin x = -j^2\sin x$
$\sin x = \sin x$

(iii) $\cos^2 x = \frac{1}{2}(1+\cos 2x)$
$\frac{(e^{jx}+e^{-jx})^2}{4} = \frac{1}{2}(1+\cos 2x)$
$\frac{e^{2jx}+e^{-2jx}+2}{4} = \frac{1}{2}(1+\cos 2x)$
$\frac{2\cos 2x+2}{4} = \frac{1}{2}(1+\cos 2x)$
$\frac{1+\cos 2x}{2} = \frac{1}{2}(1+\cos 2x)$

(iv) $\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$
$\frac{(e^{jx}-e^{-jx})}{2j}\frac{(e^{jy}-e^{-jy})}{2j} = \frac{1}{2}\frac{(e^{j(x-y)}-e^{-j(x-y)})}{2} - \frac{1}{2}\frac{(e^{j(x+y)}-e^{-j(x+y)})}{2}$
$-\frac{1}{4}(e^{jx}-e^{-jx})(e^{jy}-e^{-jy}) = \frac{1}{4}(e^{jx}e^{-jy}+e^{-jx}e^{jy}-e^{jx}e^{jy}-e^{-jx}e^{-jy})$
$-\frac{1}{4}(e^{jx}e^{jy}-e^{jx}e^{-jy}-e^{-jx}e^{jy}+e^{-jx}e^{-jy}) = \frac{1}{4}(e^{jx}e^{-jy}+e^{-jx}e^{jy}-e^{jx}e^{jy}-e^{-jx}e^{-jy})$

(v) $\sin(x+y) = \sin x\cos y + \cos x \sin y$
$\frac{e^{j(x+y)}-e^{-j(x+y)}}{2j} = \frac{e^{jx}-e^{-jx}}{2j}\cdot\frac{e^{jy}-e^{-jy}}{2} + \frac{e^{jx}+e^{-jx}}{2}\cdot\frac{e^{jy}-e^{-jy}}{2j}$
$\frac{e^{j(x+y)}-e^{-j(x+y)}}{2j} = \frac{e^{j(x+y)}+e^{j(x-y)}-e^{-j(x-y)}-e^{-j(x+y)}+e^{j(x+y)}-e^{j(x-y)}+e^{-j(x-y)}-e^{-j(x+y)}}{2\cdot 2j}$
$\frac{e^{j(x+y)}-e^{-j(x+y)}}{2j} = \frac{2e^{j(x+y)}-2e^{-j(x+y)}}{2\cdot 2j}$
$\frac{e^{j(x+y)}-e^{-j(x+y)}}{2j} = \frac{e^{j(x+y)}-e^{-j(x+y)}}{2j}$

(C) A linear system has the output $y(t) = e^{j3t}$ if the input is $x(t) = e^{j2t}$, and the output $y(t) = e^{-j3t}$ if the input is $x(t) = e^{-j2t}$. Determine the output of the system if:

(i) $x_1(t) = \cos 2t$
$e^{j2t} \Rightarrow e^{j3t}$
$e^{-j2t} \Rightarrow e^{-j3t}$
$\cos 2t = \frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t} \Rightarrow \frac{1}{2}e^{j3t} + \frac{1}{2}e^{-j3t} = \cos 3t$

(ii) $x_2(t) = \cos[2(t-1/2)]$
$e^{j2t} \Rightarrow e^{j3t}$
$e^{-j2t} \Rightarrow e^{-j3t}$
$\cos[2(t-\frac{1}{2})] = \frac{1}{2}e^{j2t}e^{-j} + \frac{1}{2}e^{-j2t}e^{j} \Rightarrow \frac{1}{2}e^{j3t}e^{-j} + \frac{1}{2}e^{-j3t}e^{j} = \cos(3t-1) = \cos[3(t-\frac{1}{3})]$
Note that even though the system is linear, it is not time-invariant, since a shift in the input of 1/2 produces a shift in the output of 1/3.

Problem 2:

(A) Consider a time-invariant system. Show that if input $x(t)$ is periodic, then output $y(t)$ must also be periodic. *Hint: Remember that periodicity is defined as $x(t-T) = x(t)$ for some period T.*

From the problem statement, $x(t) = x(t-T)$. Furthermore, since the system is time-invariant, given any $x(t) \to y(t)$, then $x(t-\tau) \to y(t-\tau)$ for any time shift τ. Choose $\tau = T$. Then $y(t) = y(t-T)$ if $x(t) = x(t-T)$ because of time-invariance. Since $x(t) = x(t-T)$ by periodicity of $x(t)$, it follows that $y(t) = y(t-T)$. Thus, $y(t)$ is periodic.

(B) Give an example of a time-invariant system and a nonperiodic input signal $x(t)$ such that the corresponding output $y(t)$ is periodic.

Consider the time-invariant system (as proven in class) $y(t) = \sin[x(t)]$. Let $x(t) = t$ (nonperiodic). Then, $y(t) = \sin t$, which is a periodic signal.

Problem 3:

(A) Consider the statement: The series interconnection of two linear time-invariant systems is a linear time-invariant system. True or false? Justify your answer.

Assume linear system S_1 such that $x(t) \to y(t)$. Assume also linear system S_2 such that $y(t) \to z(t)$. Then, choose a particular $x_1(t)$ and $x_2(t)$. By our assumption, $x_1(t) \to y_1(t) \to z_1(t)$, and $x_2(t) \to y_2(t) \to z_2(t)$. Since S_1 is linear, $ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$. Similarly, since S_2 is linear, $ay_1(t) + by_2(t) \to az_1(t) + bz_2(t)$. Hence, the series combination of S_1, S_2 yields $ax_1(t) + bx_2(t) \to az_1(t) + bz_2(t)$, which proves the combined system is linear. An identical argument proves that the series combination of S_1 and S_2 is also time-invariant.

(B) Consider the statement: The series interconnection of two nonlinear systems is a nonlinear system. True or false? Justify your answer.

Consider the systems $y(t) = x^3(t)$ and $z(t) = \sqrt[3]{y(t)}$. Both system are clearly nonlinear, yet their series interconnection yields the system $z(t) = x(t)$, which is obviously linear.

Problem 4: An even signal is a signal which satisfies the condition $x_e(-t) = x_e(t)$. An odd signal satisfies $x_o(-t) = -x_o(t)$. Any signal $x(t)$ may be decomposed into even and odd parts using $x_e(t) = [x(t) + x(-t)]/2$ and $x_o(t) = [x(t) - x(-t)]/2$.

(A) Check that the equation for $x_e(t)$ yields an even signal, and that the equation for $x_o(t)$ yields an odd signal. Also check that $x_e(t) + x_o(t) = x(t)$.

$x_e(t) = [x(t) + x(-t)]/2$
$x_e(-t) = [x(-t) + x(t)]/2 = x_e(t)$

$x_o(t) = [x(t) - x(-t)]/2$
$x_o(-t) = [x(-t) - x(t)]/2 = -[x(t) - x(-t)]/2 = -x_o(t)$

$x_e(t) + x_o(t) = [x(t) + x(-t)]/2 + [x(t) - x(-t)]/2 = [x(t) + x(-t) + x(t) - x(-t)]/2 = 2x(t)/2 = x(t)$

(B) Show that

$$\int_{-\infty}^{+\infty} x_o(t)dt = 0$$

for any odd signal $x_o(t)$.

First, break the integral in two parts:

$$\int_{-\infty}^{+\infty} x_o(t)dt = \int_{-\infty}^{0} x_o(t)dt + \int_{0}^{+\infty} x_o(t)dt$$

Let $\tau = -t$. Then $d\tau = -dt$ and

$$= -\int_{+\infty}^{0} x_o(-\tau)d\tau + \int_{0}^{+\infty} x_o(t)dt$$

Changing the limits,

$$= \int_{0}^{+\infty} x_o(-\tau)d\tau + \int_{0}^{+\infty} x_o(t)dt$$

Since $x_o(-t) = -x_o(t)$,

$$= -\int_{0}^{+\infty} x_o(\tau)d\tau + \int_{0}^{+\infty} x_o(t)dt = 0$$

(C) Show that if $x_1(t)$ is an odd signal and $x_2(t)$ is an even signal, then $x_1(t)x_2(t)$ is an odd signal.

$x_1(-t)x_2(-t) = -x_1(t) \cdot x_2(t) = -x_1(t)x_2(t)$

(D) Show that

$$\int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} x_e^2(t)dt + \int_{-\infty}^{+\infty} x_o^2(t)dt$$

where $x_e(t)$ and $x_o(t)$ are the even and odd parts of $x(t)$.

Express $x(t)$ as the sum of its even and odd parts:

$$\begin{aligned}\int_{-\infty}^{+\infty} x^2(t)dt &= \int_{-\infty}^{+\infty} [x_e(t) + x_o(t)]^2 dt = \int_{-\infty}^{+\infty} [x_e^2(t) + x_o^2(t) + 2x_e(t)x_o(t)]dt \\ &= \int_{-\infty}^{+\infty} x_e^2(t)dt + \int_{-\infty}^{+\infty} x_o^2(t)dt + 2\int_{-\infty}^{+\infty} x_e(t)x_o(t)dt\end{aligned}$$

But $x_e(t)x_o(t)$ yields an odd function such that

$$2\int_{-\infty}^{+\infty} x_e(t)x_o(t)dt = 0$$

Therefore,

$$\int_{-\infty}^{+\infty} x^2(t)dt = \int_{-\infty}^{+\infty} x_e^2(t)dt + \int_{-\infty}^{+\infty} x_o^2(t)dt$$