Inter American University of Puerto Rico
Bayamón Campus
School of Engineering
Department of Electrical Engineering

ELEN 3301 – Electric Circuits I

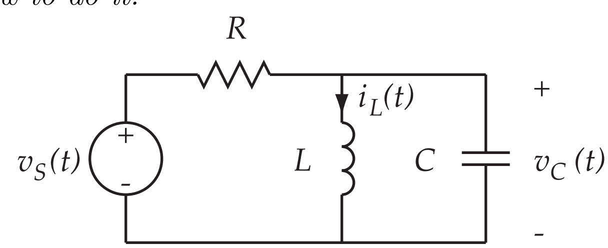
Problem Set 9 Solutions

Due Wednesday, November 10

Reminder: Quiz 2 will be offered Wednesday, November 10 and will cover material given in class until October 27.

Problem 1:

(A) For the circuit below, $v_C(0) = 0$, $i_L(0) = 0$ and $v_S = V_S u(t)$. Find an algebraic expression for $v_C(t)$ if $R < \frac{1}{2}\sqrt{\frac{L}{C}}$. *You don't have to sketch the solution, but make sure you understand how to do it.*



KCL at the top-right node yields:

$$\frac{v_S - v_C}{R} = i_L + C\frac{dv_C}{dt}$$

KVL at the right loop yields:

$$v_C = L\frac{di_L}{dt} \Rightarrow \frac{di_L}{dt} = \frac{v_C}{L}$$

In order to substitute, we derive the first equation:

$$\frac{1}{R}\frac{dv_S}{dt} - \frac{1}{R}\frac{dv_C}{dt} = \frac{di_L}{dt} + C\frac{d^2v_C}{dt^2} = \frac{v_C}{L} + C\frac{d^2v_C}{dt^2}$$

In standard form:

$$\frac{d^2 v_C}{dt^2} + \frac{1}{RC}\frac{dv_C}{dt} + \frac{v_C}{LC} = \frac{1}{RC}\frac{dv_S}{dt}$$

Note that v_S appears with a derivative, which is harder to solve. An easier way is to solve for another variable and find v_C using this variable. The easiest variable to find is i_L. Starting from the first two equations, substitute $v_C = L\frac{di_L}{dt}$ into the KCL equation:

$$\frac{v_S}{R} - \frac{L}{R}\frac{di_L}{dt} = i_L + LC\frac{d^2 i_L}{dt^2}$$

In standard form:

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{v_S}{RLC}$$

The particular solution is $i_{LP} = \frac{V_S}{R}$. Let $\alpha = \frac{1}{2RC}$ and $w_0 = \frac{1}{\sqrt{LC}}$. Since $R < \frac{1}{2}\sqrt{\frac{L}{C}}$, $w_0 < \alpha$ and the homogeneous solution takes the form

$$i_{LH} = Ae^{s_1 t} + Be^{s_2 t} \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

The initial conditions are $i_L(0) = 0$ and $\frac{di_L}{dt}\Big|_{t=0} = \frac{v_C(0)}{L} = 0$, which yield the equations

$$A + B + \frac{V_S}{R} = 0 \qquad As_1 + Bs_2 = 0$$
$$\Rightarrow A = \frac{V_S}{R} \cdot \frac{s_2}{s_1 - s_2} \qquad B = -\frac{V_S}{R} \cdot \frac{s_1}{s_1 - s_2}$$

Since $v_C = L\frac{di_L}{dt}$,

$$v_C = LAs_1 e^{s_1 t} + LBs_2 e^{s_2 t} = \frac{LV_S}{R} \cdot \frac{s_1 s_2}{s_1 - s_2} e^{s_1 t} - \frac{LV_S}{R} \cdot \frac{s_1 s_2}{s_1 - s_2} e^{s_2 t}$$

(B) Repeat part (A) if $R > \frac{1}{2}\sqrt{\frac{L}{C}}$.

The method is identical to the one above, except that since $R > \frac{1}{2}\sqrt{\frac{L}{C}}$, $w_0 > \alpha$ and the homogeneous solution takes the form

$$i_{LH} = Ae^{-\alpha t}\cos(\omega_d t) + Be^{-\alpha t}\sin(\omega_d t) \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

The initial conditions yield the equations

$$A + \frac{V_S}{R} = 0 \qquad -\alpha A + \omega_d B = 0$$

$$\Rightarrow A = -\frac{V_S}{R} \qquad B = -\frac{\alpha}{\omega_d} \cdot \frac{V_S}{R}$$

Since $v_C = L\frac{di_L}{dt}$,

$$v_C = \frac{LV_S}{R} \cdot \frac{\omega_0^2}{\omega_d} e^{-\alpha t} \sin(\omega_d t) = \frac{V_S}{RC\omega_d} e^{-\alpha t} \sin(\omega_d t)$$

Problem 2: Find the solution for the differential equations listed below with the given initial conditions. *Again, you don't have to sketch the solutions, but make sure you understand how to do it.*

(A) $\dfrac{dv}{dt} + \dfrac{v}{\tau} = \dfrac{V_S u(t)}{\tau} \qquad v(0) = 0$

$$v = V_S(1 - e^{-t/\tau})$$

(B) $\dfrac{d^2v}{dt^2} + 2\alpha\dfrac{dv}{dt} + \omega_0^2 v = \omega_0^2 V_S u(t) \qquad \omega_0 > \alpha \qquad v(0) = 0 \qquad \left.\dfrac{dv}{dt}\right|_{t=0} = 0$

$$v = V_S - V_S e^{-\alpha t} \cos(\omega_d t) - \frac{\alpha}{\omega_d} V_S e^{-\alpha t} \sin(\omega_d t) \qquad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

(C) $\dfrac{d^2v}{dt^2} + 2\alpha\dfrac{dv}{dt} + \omega_0^2 v = \omega_0^2 V_S u(t) \qquad \omega_0 < \alpha \qquad v(0) = 0 \qquad \left.\dfrac{dv}{dt}\right|_{t=0} = 0$

$$v = V_S + V_S \frac{s_2}{s_1 - s_2} e^{s_1 t} - V_S \frac{s_1}{s_1 - s_2} e^{s_2 t} \qquad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

Problem 3: Review problem sets 5-8 *carefully.* Summarize all the circuits, differential equations, solutions and graphs given to make sure you understand clearly all the differences between all types of circuits. *You do not have to hand in anything for this problem.*