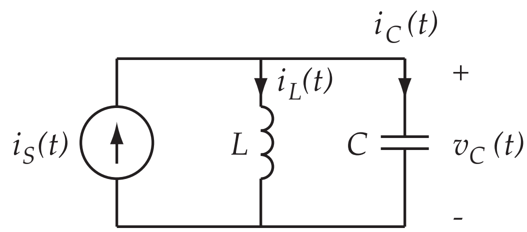
Inter American University of Puerto Rico
Bayamón Campus
School of Engineering
Department of Electrical Engineering

ELEN 3301 – Electric Circuits I

Problem Set 7 Solutions

Due Wednesday, October 20

Problem 1: For the circuit below, $v_C(0) = 0$ and $i_L(0) = 0$.



(A) Let $i_S(t) = I_S u(t)$. Find an algebraic expression and sketch $i_L(t)$ for $-\pi\sqrt{LC} < t < 2\pi\sqrt{LC}$.

To derive the differential equation, use KCL in the top node:

$$i_S = i_C + i_L$$

Note that $i_C = C\frac{dv_C}{dt}$ and $v_C = v_L = L\frac{di_L}{dt}$. Substituting these expressions yields

$$i_S = C\frac{dv_C}{dt} + i_L = LC\frac{d^2 i_L}{dt^2} + i_L$$

Divide by LC to leave the equation in standard form

$$\frac{d^2 i_L}{dt^2} + \frac{i_L}{LC} = \frac{i_S}{LC}$$

For $t < 0$, $i_S = 0$ and $i_L = 0$. For $t > 0$, $i_S = I_S$. The differential equation in this period is

$$\frac{d^2 i_L}{dt^2} + \frac{i_L}{LC} = \frac{I_S}{LC}$$

By now, it should be fairly obvious that the particular solution is $i_L = I_S$, which corresponds to a steady state as $t \to \infty$ where the inductor acts as a short, the capacitor as an open, and all the current I_S flows through the inductor. From Problem Set 6, recall that the homogeneous solution for i_L is of the form

$$i_{L,Homogeneous}(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

where $w_0 = \frac{1}{\sqrt{LC}}$. The general solution is

$$i_L(t) = I_S + A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

The initial condition $i_L(0) = 0$ yields

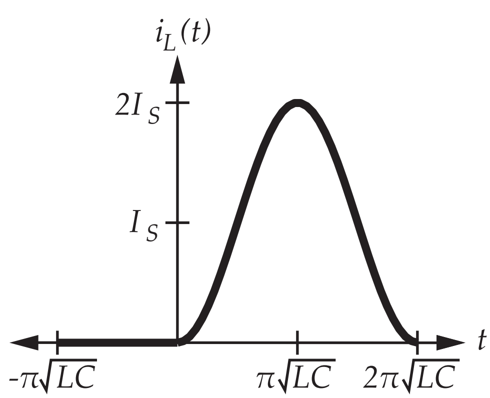
$$i_L(0) = I_S + B = 0 \Rightarrow B = -I_S$$

The initial condition $v_C(0) = 0$ implies that $\frac{di_L}{dt} = 0$ since $\frac{di_L}{dt} = \frac{v_C}{L}$. In order to apply this initial condition, the derivative of i_L is computed and evaluated at zero:

$$\left.\frac{di_L}{dt}\right|_{t=0} = A\omega_0 = 0 \Rightarrow A = 0$$

Thus, the solution is

$$\begin{aligned} i_L(t) &= 0 && t < 0 \\ i_L(t) &= I_S[1 - \cos(\omega_0 t)] \quad \omega_0 = \frac{1}{\sqrt{LC}} && t > 0 \end{aligned}$$

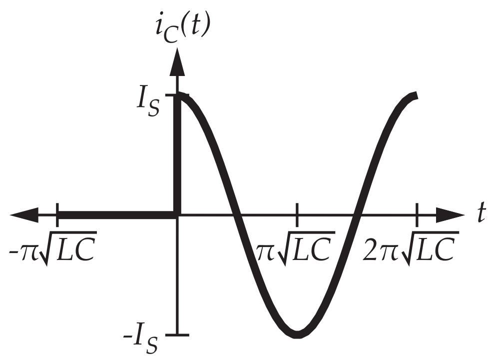


(B) Let $i_S(t) = I_S u(t)$. Find an algebraic expression and sketch $i_C(t)$ for $-\pi\sqrt{LC} < t < 2\pi\sqrt{LC}$. *Hint: Use KCL and the expression for $i_L(t)$ found in part (A).*

Note that $i_C = i_S - i_L$. Thus,

$$i_C(t) = 0 \qquad t < 0$$

$$i_C(t) = I_S - I_S[1 - \cos(\omega_0 t)] = I_S \cos(\omega_0 t) \quad \omega_0 = \frac{1}{\sqrt{LC}} \qquad t > 0$$



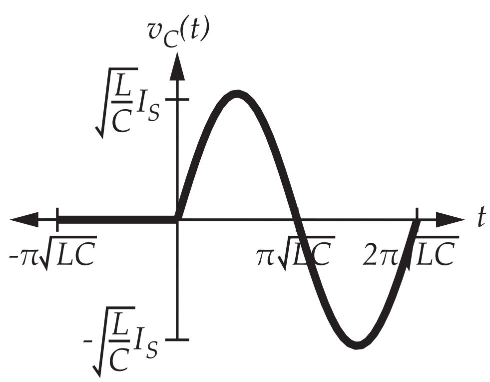
(C) Let $i_S(t) = I_S u(t)$. Find an algebraic expression and sketch $v_C(t)$ for $-\pi\sqrt{LC} < t < 2\pi\sqrt{LC}$. *Hint: Use KVL and the expression for $i_L(t)$ found in part (A).*

Note that $v_C = v_L = L\frac{di_L}{dt}$. Thus,

$$v_C(t) = 0 \qquad t < 0$$

$$v_C(t) = L\frac{d}{dt}\Big(I_S[1 - \cos(\omega_0 t)]\Big) = L I_S \omega_0 \sin(\omega_0 t) = I_S\sqrt{\frac{L}{C}}\sin(\omega_0 t) \qquad t > 0$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$.



(D) Let $i_S(t) = Q\delta(t)$. Repeat parts (A), (B) and (C) using this new input. Remember to sketch all three responses.

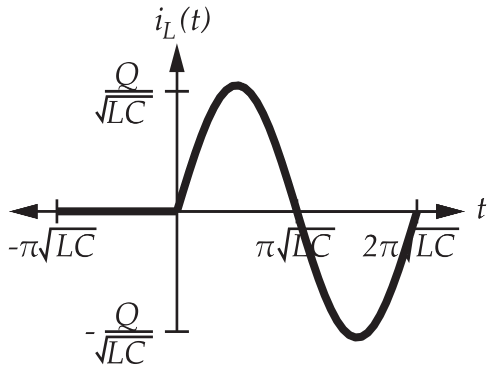
Note that

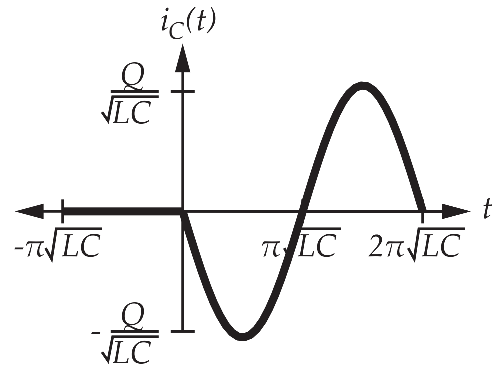
$$Q\delta(t) = \frac{Q}{I_S}\frac{d}{dt}I_S u(t)$$

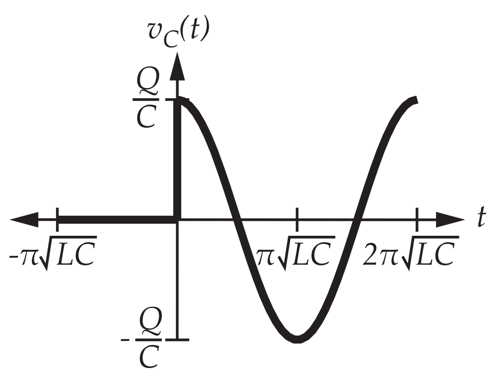
Therefore, because of linearity, the application of the same operator to the answers obtained in the previous three parts yields the responses to the new input:

$$\begin{aligned}
i_L(t) &= \frac{Q}{I_S}\frac{d}{dt}I_S[1-\cos(\omega_0 t)] = Q\omega_0\sin(\omega_0 t) = \frac{Q}{\sqrt{LC}}\sin(\omega_0 t)\\
i_C(t) &= \frac{Q}{I_S}\frac{d}{dt}I_S\cos(\omega_0 t) = -\frac{Q}{\sqrt{LC}}\sin(\omega_0 t)\\
v_C(t) &= \frac{Q}{I_S}\frac{d}{dt}I_S\sqrt{\frac{L}{C}}\sin(\omega_0 t) = \frac{Q}{C}\cos(\omega_0 t)
\end{aligned}$$

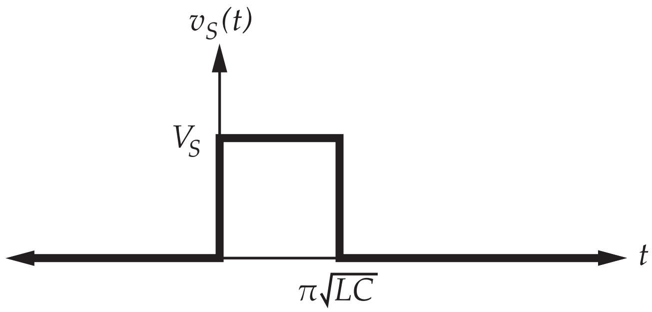
where $\omega_0 = \frac{1}{\sqrt{LC}}$ in all cases.

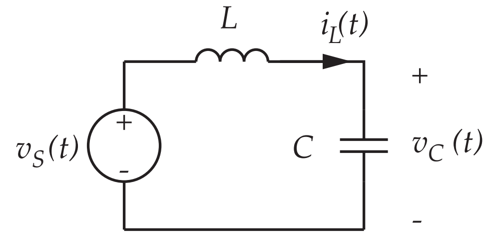






Problem 2: For the circuit below, $v_C(0) = 0$ and $i_L(0) = 0$. If $v_S(t)$ is as shown in the figure below, find an algebraic expression and sketch $v_C(t)$ for $-\pi\sqrt{LC} < t < 3\pi\sqrt{LC}$.





To derive the differential equation, use KVL:

$$v_S = v_L + v_C$$

Note that $v_L = L\frac{di_L}{dt}$ and $i_L = C\frac{dv_C}{dt}$. Substituting these expressions yields

$$v_S = L\frac{di_L}{dt} + v_C = LC\frac{d^2v_C}{dt^2} + v_C$$

Divide by LC to leave the equation in standard form

$$\frac{d^2v_C}{dt^2} + \frac{v_C}{LC} = \frac{v_S}{LC}$$

For $t < 0$, $v_S = 0$ and $v_C = 0$. For $0 < t < \pi\sqrt{LC}$, $v_S = V_S$. The differential equation in this period is

$$\frac{d^2v_C}{dt^2} + \frac{v_C}{LC} = \frac{V_S}{LC}$$

The particular solution is $v_C = V_S$, which corresponds to a steady state as $t \to \infty$ where the inductor acts as a short, the capacitor as an open, and all the voltage V_S shows across the capacitor. From Problem Set 6, recall that the homogeneous solution for v_C is of the form

$$v_{C,Homogeneous}(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

where $w_0 = \frac{1}{\sqrt{LC}}$. The general solution is

$$v_C(t) = V_S + A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

The initial condition $v_C(0) = 0$ yields

$$v_C(0) = V_S + B = 0 \Rightarrow B = -V_S$$

The initial condition $i_L(0) = 0$ implies that $\frac{dv_C}{dt} = 0$ since $\frac{dv_C}{dt} = \frac{i_L}{C}$. In order to apply this initial condition, the derivative of v_C is computed and evaluated at zero:

$$\left.\frac{dv_C}{dt}\right|_{t=0} = A\omega_0 = 0 \Rightarrow A = 0$$

Thus, the solution for $0 < t < \pi\sqrt{LC}$ is

$$v_C(t) = V_S[1 - \cos(\omega_0 t)] \quad \omega_0 = \frac{1}{\sqrt{LC}} \qquad\qquad 0 < t < \pi\sqrt{LC}$$

For $t > \pi\sqrt{LC}$, $v_S = 0$. The differential equation in this period is

$$\frac{d^2 v_C}{dt^2} + \frac{v_C}{LC} = 0$$

The initial conditions must be obtained by evaluating the expression obtained for $v_C(t)$ at $t = \pi\sqrt{LC}$. Thus,

$$\begin{aligned} v_C(\pi\sqrt{LC}) &= 2V_S \\ \frac{dv_C}{dt} &= \frac{i_L(t)}{C} = V_S\omega_0 \sin(\omega_0 t) \\ \left.\frac{dv_C}{dt}\right|_{t=\pi\sqrt{LC}} &= \frac{i_L(\pi\sqrt{LC})}{C} = 0 \end{aligned}$$

The particular solution is $v_C = 0$ for this period. Thus, the general solution reduces to the homogeneous solution:

$$v_C(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

The initial condition $v_C(\pi\sqrt{LC}) = 2V_S$ yields

$$v_C(\pi\sqrt{LC}) = -B = 2V_S \Rightarrow B = -2V_S$$

In order to apply the initial condition $\frac{dv_C}{dt}|_{t=\pi\sqrt{LC}} = 0$, the derivative of v_C is computed and evaluated at zero:

$$\left.\frac{dv_C}{dt}\right|_{t=\pi\sqrt{LC}} = -A\omega_0 = 0 \Rightarrow A = 0$$

Thus, the solution for $t > \pi\sqrt{LC}$ is

$$v_C(t) = -2V_S\cos(\omega_0 t) \quad \omega_0 = \frac{1}{\sqrt{LC}} \qquad\qquad t > \pi\sqrt{LC}$$

The complete solution is

$$
\begin{aligned}
&v_C(t) = 0 && t < 0 \\
&v_C(t) = V_S[1 - \cos(\omega_0 t)] \quad \omega_0 = \frac{1}{\sqrt{LC}} && 0 < t < \pi\sqrt{LC} \\
&v_C(t) = -2V_S\cos(\omega_0 t) \quad \omega_0 = \frac{1}{\sqrt{LC}} && t > \pi\sqrt{LC}
\end{aligned}
$$