Inter American University of Puerto Rico
Bayamón Campus
School of Engineering
Department of Electrical Engineering

ELEN 3301 – Electric Circuits I

Problem Set 6 Solutions

Due Wednesday, October 13

Problem 1: Show that

$$v(t) = Ke^{-\alpha t} \sin(\omega_d t + \phi)$$

is the general solution of the differential equation

$$\frac{d^2v}{dt^2} + 2\alpha \frac{dv}{dt} + \omega_0^2 v = 0 \quad \text{where} \quad \omega_0 > \alpha$$

if $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$. The method will be outlined in the following parts.

(A) Show that

$$\frac{dv}{dt} = -\alpha K e^{-\alpha t} \sin(\omega_d t + \phi) + \omega_d K e^{-\alpha t} \cos(\omega_d t + \phi)$$

Use the multiplication rule of derivatives:

$$\frac{d}{dt}[f(t)g(t)] = \frac{df}{dt} g(t) + f(t) \frac{dg}{dt}$$

(B) Show that

$$\frac{d^2v}{dt^2} = \alpha^2 K e^{-\alpha t} \sin(\omega_d t + \phi) - 2\alpha\omega_d K e^{-\alpha t} \cos(\omega_d t + \phi) - \omega_d^2 K e^{-\alpha t} \sin(\omega_d t + \phi)$$

Use the multiplication rule again, starting from $\frac{dv}{dt}$.

(C) Substitute the expressions for $\frac{d^2v}{dt^2}$, $\frac{dv}{dt}$ and v into the differential equation. Group the coefficients of the sine and cosine terms and equate each group to zero. You should be left with two algebraic equations. Show that the equations hold if $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$.

sine equation:

$$\begin{aligned}
\alpha^2 - \omega_d^2 + 2\alpha(-\alpha) + \omega_0^2 &= 0 \\
\alpha^2 - \omega_d^2 - 2\alpha^2 + \omega_0^2 &= 0 \\
-\omega_d^2 - \alpha^2 + \omega_0^2 &= 0 \\
\omega_0^2 - \alpha^2 &= \omega_d^2 \\
\omega_d &= \sqrt{\omega_0^2 - \alpha^2} \quad \text{(condition on } \omega_d\text{)}
\end{aligned}$$

cosine equation:

$$\begin{aligned}
-2\alpha\omega_d + 2\alpha(\omega_d) &= 0 \\
-2\alpha\omega_d + 2\alpha\omega_d &= 0 \\
0 &= 0
\end{aligned}$$

(D) You have found the general solution for the differential equation given in the problem statement. Given this solution, find the solution for the differential equation

$$\frac{d^2v}{dt^2} + \omega_0^2 v = 0$$

This equation is the special case of the original equation when $\alpha = 0$. Thus, the solution must also work in this equation if we let $\alpha = 0$. Note that $w_d = w_0$ and $e^{-\alpha t} = 1$ in this case. Thus,

$$v(t) = K\sin(\omega_0 t + \phi)$$

(E) Show that the solution for the differential equation found in the previous part can be converted to the form

$$v(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$$

Hint: $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

Using

$$v(t) = K\sin(\omega_0 t + \phi)$$

let $x = \omega_0 t$ and $y = \phi$. Then,

$$v(t) = K[\sin(\omega_0 t)\cos(\phi) + \cos(\omega_0 t)\sin(\phi)]$$

$$v(t) = K\cos(\phi)\sin(\omega_0 t) + K\sin(\phi)\cos(\omega_0 t)$$

which has the form $v(t) = A\sin(\omega_0 t) + B\cos(\omega_0 t)$ if $A = K\cos(\phi)$ and $B = K\sin(\phi)$.