Inter American University of Puerto Rico
Bayamón Campus
School of Engineering
Department of Electrical Engineering

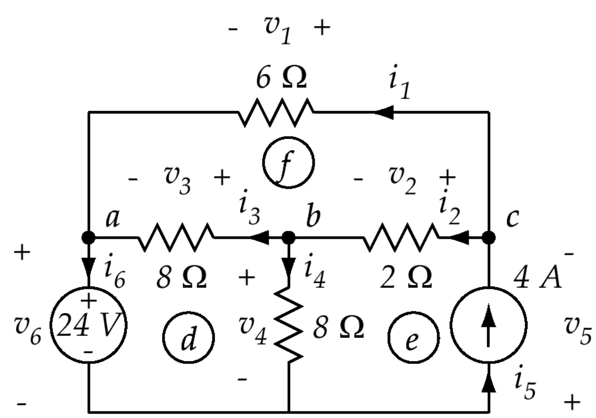
ELEN 3301 – Electric Circuits I

Problem Set 1 Solutions

Due Wednesday, August 25

Problem 1:

(A) Solve for the current and voltages of every branch in the following circuit:



The circuit has 6 currents and 6 voltages, for a total of 12 unknowns. The 12 equations required to solve the system are:

Device laws: $v_1 = 6\Omega \cdot i_1$ $\qquad$ $v_2 = 2\Omega \cdot i_2$

$v_3 = 8\Omega \cdot i_3$ $\qquad$ $v_4 = 8\Omega \cdot i_4$

$i_5 = 4A$ $\qquad$ $v_6 = 24V$

KCL at node a: $i_1 + i_3 = i_6$ $\qquad$ KCL at node b: $i_4 + i_3 = i_2$

KCL at node c: $i_1 + i_2 = i_5$ $\qquad$ KVL at loop d: $v_6 + v_3 = v_4$

KVL at loop e: $v_4 + v_5 + v_2 = 0$ $\qquad$ KVL at loop f: $v_3 + v_2 = v_1$

The resulting values are:

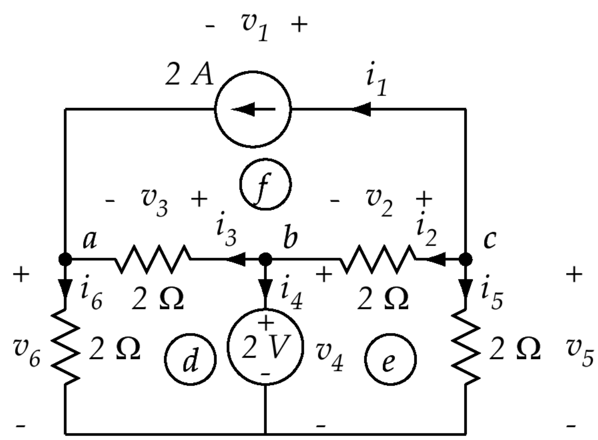
$v_1 = 6V$	$v_2 = 6V$	$v_3 = 0V$	$v_4 = 24V$
$v_5 = -30V$	$v_6 = 24V$	$i_1 = 1A$	$i_2 = 3A$
$i_3 = 0A$	$i_4 = 3A$	$i_5 = 4A$	$i_6 = 1A$

(B) Check that power is conserved by computing the power in each branch and adding all the branches.

$$v_1 i_1 = 6W \qquad v_2 i_2 = 18W$$
$$v_3 i_3 = 0W \qquad v_4 i_4 = 72W$$
$$v_5 i_5 = -120W \qquad v_6 i_6 = 24W$$

The total power in the circuit is: $6W + 18W + 0W + 72W - 120W + 24W = 0$. Note that the current source is providing all the power for the circuit, while the voltage source is dissipating power, as if it were a resistor.

(C) Repeat parts (A) and (B) for the following circuit:



$$\text{Device laws: } i_1 = 2A \qquad v_2 = 2\Omega \cdot i_2$$
$$v_3 = 2\Omega \cdot i_3 \qquad v_4 = 2V$$
$$v_5 = 2\Omega \cdot i_5 \qquad v_6 = 2\Omega \cdot i_6$$

KCL at node a: $i_1 + i_3 = i_6$ KCL at node b: $i_4 + i_3 = i_2$

KCL at node c: $i_1 + i_2 + i_5 = 0$ KVL at loop d: $v_6 + v_3 = v_4$

KVL at loop e: $v_4 + v_2 = v_5$ KVL at loop f: $v_3 + v_2 = v_1$

The resulting values are:

$$v_1 = -4V \qquad v_2 = -3V \qquad v_3 = -1V \qquad v_4 = 2V$$
$$v_5 = -1V \qquad v_6 = 3V \qquad i_1 = 2A \qquad i_2 = -1.5A$$
$$i_3 = -0.5A \qquad i_4 = -1A \qquad i_5 = -0.5A \qquad i_6 = 1.5A$$

$$v_1 i_1 = -8W \qquad v_2 i_2 = 4.5W$$
$$v_3 i_3 = 0.5W \qquad v_4 i_4 = -2W$$
$$v_5 i_5 = 0.5W \qquad v_6 i_6 = 4.5W$$

The total power in the circuit is: $-8W + 4.5W + 0.5W - 2W + 0.5W + 4.5W = 0$.

(D) Now, multiply the voltages you obtained in part (A) by the currents you obtained in part (C), associating corresponding branches and add the result. Can you explain this result?

Voltages from the first circuit:

$$v_1 = 6V \quad v_2 = 6V \quad v_3 = 0V \quad v_4 = 24V \quad v_5 = -30V \quad v_6 = 24V$$

Currents from the second circuit:

$$i_1 = 2A \quad i_2 = -1.5A \quad i_3 = -0.5A \quad i_4 = -1A \quad i_5 = 0.5A \quad i_6 = 1.5A$$

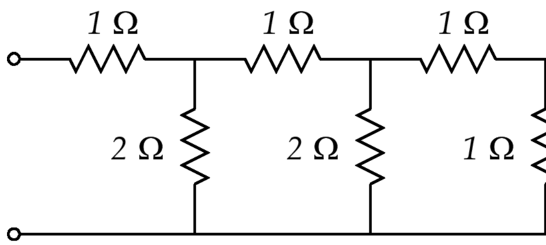
Note that i_5 in the second circuit was reversed to make it correspond to voltage v_5 in the first circuit.

$$\begin{aligned} v_1 i_1 &= 12W & v_2 i_2 &= -9W \\ v_3 i_3 &= 0W & v_4 i_4 &= -24W \\ v_5 i_5 &= -15W & v_6 i_6 &= 36W \end{aligned}$$

The total is: $12W - 9W + 0W - 24W - 15W + 36W = 0$. This result is known as *Tellegen's theorem* which states that, given a network, any set of values that fulfills KVL multiplied by any other corresponding set of values that fulfills KCL will always be zero. Note that the values don't have to be voltages and currents, as long as they fulfill KVL or KCL. Also note that nothing is said about the elements in the network; this theorem applies even if the elements are nonlinear. This theorem can be proven using only KVL and KCL. Furthermore, using this theorem and KVL or KCL, the remaining law, KCL or KVL can be derived. Thus, this theorem is of equal mathematical importance in network analysis as KVL or KCL.

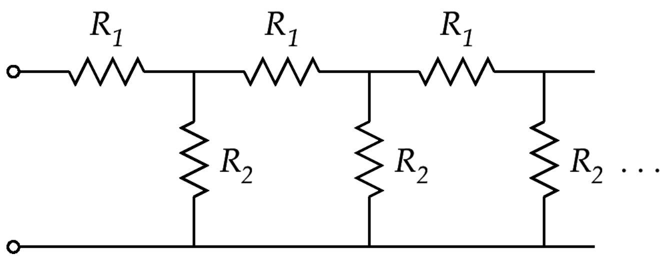
Problem 2:

(A) Evaluate the resistance between the terminals of the ladder circuit shown below:



$$R_{eq} = 1\Omega + \left[2\Omega || \{1\Omega + 2\Omega || (1\Omega + 1\Omega)\}\right] = 2\Omega$$

(B) Generalize the equivalent resistance of an infinite ladder circuit:



Note that since the ladder is infinite, we can remove the ladder *after* the first section containing R_1 and R_2, and replace it with the same equivalent resistance of the full ladder. Thus, $R_{eq} = R_1 + R_2 || R_{eq}$. Solving this equation for R_{eq} and discarding the negative root, we obtain:

$$R_{eq} = \frac{R_1 + \sqrt{R_1(R_1 + 4R_2)}}{2}$$

(C) Evaluate the result obtained in the previous part for the following cases. For cases (ii) and (iii), interpret the result by drawing a circuit which represents the given condition.

(i) $2R_1 = R_2$
$R_{eq} = 2R_1$

(ii) $R_1 >> R_2$
In this case, $R_1 + 4R_2 \approx R_1$, and $R_{eq} = R_1$. This condition corresponds to R_2 becoming a short circuit, so that the rest of the ladder disappear and the terminals can only "see" the first R_1.

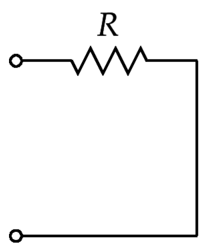
(iii) $R_1 << R_2$

In this case, $R_1 + 4R_2 \approx 4R_2$, $R_1 + 2\sqrt{R_1 R_2} \approx 2\sqrt{R_1 R_2}$, and $R_{eq} = \sqrt{R_1 R_2}$. This condition corresponds to R_1 becoming almost a short circuit, so that only the first R_1 is visible. However, the rest of R_2's in the ladder are in parallel, so that their combined value tends to zero. Thus, we cannot discard neither the first R_1, nor the infinite ladder of parallel R_2's. A geometric mean seems a good solution to take into account both diminishing values. Although the interpretation in this case is less satisfying than in the previous case, at least the result should make some sense in light of this interpretation.

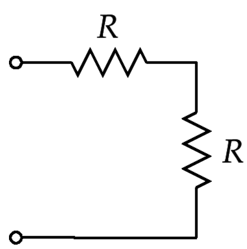
(iv) $R_1 = R_2$

$$R_{eq} = \frac{1+\sqrt{5}}{2} R_1$$

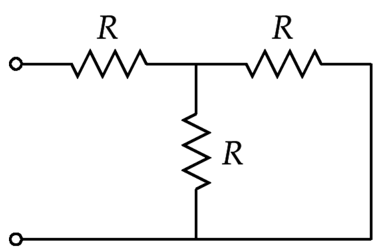
(D) Can you recognize the number in condition (iv) of the previous part? Evaluate the equivalent resistance for each circuit in the series below. Can you identify this sequence? Can you relate the number in condition (iv) to this sequence?



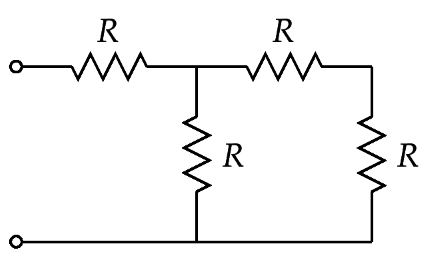
(a)



(b)



(c)



(d)

The number in condition (iv) is the *Golden ratio* of the greeks, believed to form the most aesthetically pleasing rectangles. It also has a lot of connections in nature. The sequence is: R, $2R$, $\frac{3}{2}R$, and $\frac{5}{3}R$. These numbers are ratios of the *Fibonacci sequence*, 0,1,1,2,3,5,8,13,21..., where $x_n = x_{n-1} + x_{n-2}$, $x_0 = 0$ and $x_1 = 1$. Since it can be seen that the networks are approching the infinite ladder with $R = R_1 = R_2$, it can be guessed that the ratio of consecutive Fibonacci numbers converges to the Golden mean, which is, in fact, the case!