Inter American University of Puerto Rico
Bayamón Campus
School of Engineering
Department of Electrical Engineering

ELEN 3301 – Electric Circuits I

Problem Set 11 Solutions

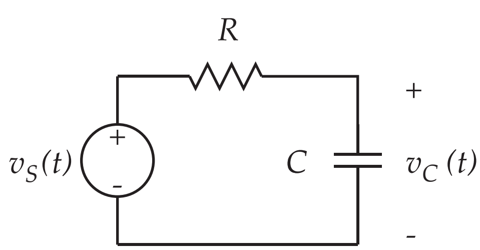
Due Wednesday, December 8

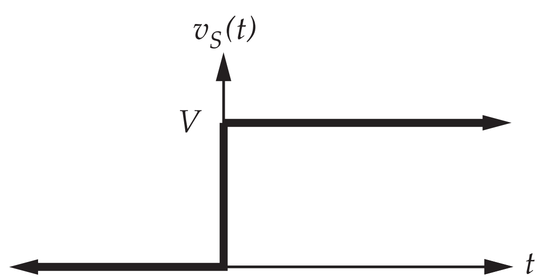
Problem 1: The purpose of this problem is to improve the efficiency of the circuit shown in class by charging it in two separate steps instead of in one step only.

The circuit below with $v_C(0) = 0$ was shown to deliver an amount of energy $\frac{1}{2}CV^2$ to a capacitor C through a resistor R. However, the amount of energy lost in the resistor R is equal to the amount of energy delivered, $\frac{1}{2}CV^2$. Thus, the voltage source has to provide CV^2 of energy in the process. Thus, the efficiency of charging a capacitor through a resistor in one step is, at best,

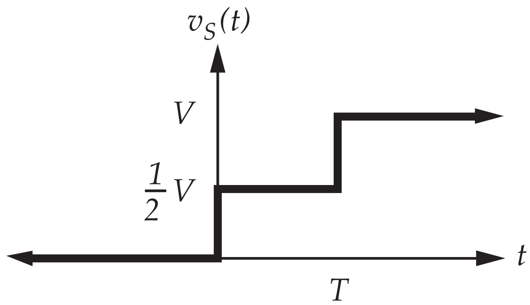
$$\frac{\frac{1}{2}CV^2}{CV^2} = 50\%$$

(where efficiency is defined as the ratio of the energy stored in the capacitor to the energy delivered by the source).





Instead of charging the capacitor to a voltage V in one step, as done with the input shown above, assume the input steps to $\frac{1}{2}V$ and stays at this level for a period $T >> RC$, as shown below. At the end of this step, the capacitor will be charged to $\frac{1}{2}V$. Then, at $t = T$, the source voltage steps again from $\frac{1}{2}V$ to V in order to charge the capacitor from $\frac{1}{2}V$ to V. By the end of the problem, you will show that the overall efficiency of charging the capacitor using this input is increased to 67%.



(A) Analyze the circuit for $0 < t < T$. Recall that $T >> RC$, so that you may assume that all transients have died (i.e., all the exponentials have disappeared). Note that in this period, $v_S(t) = \frac{1}{2}V$. Begin by computing:

(i) the amount of energy stored in the capacitor at the end of this period ($t = T^-$).

$$E_C(T^-) = \frac{1}{2}C\left(\frac{1}{2}V\right)^2 = \frac{1}{8}CV^2$$

(ii) the amount of energy dissipated in the resistor during this period ($0 < t < T$).

$$E_R(T^-) = E_C(T^-) = \frac{1}{2}C\left(\frac{1}{2}V\right)^2 = \frac{1}{8}CV^2$$

(iii) the amount of energy delivered by the source during this period ($0 < t < T$).

$$E_S(T^-) = -2E_C(T^-) = -2E_R(T^-) = -C\left(\frac{1}{2}V\right)^2 = -\frac{1}{4}CV^2$$

(Hint: You may use the results obtained in class.)

(B) Show that

$$\int_0^\infty e^{-\frac{t}{\tau}}dt = \tau$$

$$\int_0^\infty e^{-\frac{t}{\tau}}dt = -\tau e^{-t/\tau}\Big]_0^\infty = -\tau e^{-\infty/\tau} - (-\tau e^{-0/\tau}) = 0 - (-\tau) = \tau$$

(C) Analyze the circuit from $t > T$. For simplicity, convince yourself that the origin in time can be shifted to time T, such that the voltage source $v_S(t)$ goes from 0 to $\frac{1}{2}V$ at time $t = -T$ and goes from $\frac{1}{2}V$ to V at time $t = 0$. The problem now is the same as the one worked in class, except that $v_C(0) = \frac{1}{2}V$. Repeat the problem as worked in class with this new initial condition, that is, compute *directly* (i.e., find the vi product and integrate):

(i) the amount of energy delivered to the capacitor when $t \to \infty$. *Remember that the initial energy stored in the capacitor is* not *zero.*

$$E_C(0) = \frac{1}{8}CV^2$$

$$E_C(\infty) - E_C(0) = \frac{1}{2}CV^2 - \frac{1}{8}CV^2 = \frac{3}{8}CV^2$$

(ii) the amount of energy dissipated in the resistor during this period.

$$v_C(t) = \frac{1}{2}Ve^{-\frac{t}{RC}} + V(1 - e^{-\frac{t}{RC}}) = V + V\Big(\frac{1}{2} - 1\Big)e^{-\frac{t}{RC}} = V - \frac{V}{2}e^{-\frac{t}{RC}}$$

$$v_R(t) = V - v_C(t) = \frac{V}{2}e^{-\frac{t}{RC}} \qquad i(t) = \frac{v_R(t)}{R} = \frac{V}{2R}e^{-\frac{t}{RC}}$$

$$E_R(0 < t < \infty) = \int_0^\infty \frac{V^2}{4R}e^{-\frac{2t}{RC}} = \frac{V^2}{4R} \cdot \frac{RC}{2} = \frac{1}{8}CV^2$$

(iii) the amount of energy delivered by the source during this period.

$$E_S(0 < t < \infty) = \int_0^\infty -\frac{V^2}{2R}e^{-\frac{t}{RC}} = -\frac{V^2}{2R} \cdot RC = -\frac{1}{2}CV^2$$

(iv) Verify that energy is conserved.

$$E_C(\infty) - E_C(0) + E_R(0 < t < \infty) + E_S(0 < t < \infty) = \frac{3}{8}CV^2 + \frac{1}{8}CV^2 - \frac{1}{2}CV^2 = 0$$

(Hint: The result of the previous part can be helpful in the solution of this problem.)

(D) Combine the results of parts (A) and (C) by adding all the energy supplied by the source and all the energy delivered to the capacitor to show that the overall efficiency is increased to 67%. Also add all the energy dissipated in the resistor to verify that energy has been conserved.

	E_S	E_R	E_C	Efficiency $\{E_C/(-E_S)\}$
1st Stage $(0 < t < T)$	$-\frac{1}{4}CV^2$	$\frac{1}{8}CV^2$	$\frac{1}{8}CV^2$	$\frac{\frac{1}{8}CV^2}{\frac{1}{4}CV^2} = 50\%$
2nd Stage $(T < t < \infty)$	$-\frac{1}{2}CV^2$	$\frac{1}{8}CV^2$	$\frac{3}{8}CV^2$	$\frac{\frac{3}{8}CV^2}{\frac{1}{2}CV^2} = 75\%$
Total $(0 < t < \infty)$	$-\frac{3}{4}CV^2$	$\frac{1}{4}CV^2$	$\frac{1}{2}CV^2$	$\frac{\frac{1}{2}CV^2}{\frac{3}{4}CV^2} = 66\%$

Overall conservation of energy:

$$E_C(\infty) + E_R(0 < t < \infty) + E_S(0 < t < \infty) = \frac{1}{2}CV^2 + \frac{1}{4}CV^2 - \frac{3}{4}CV^2 = 0$$