

# Math 53 Lecture: The First Derivative Test

Lecturer: Jose Maria L. Esaner IV, Ph.D.  
Lecture 9

**Definition 1** A function  $f$  defined on an interval is increasing (decreasing) on the interval if and only if

$$f(x_1) < f(x_2) \quad (f(x_1) < f(x_2)) \quad \text{whenever } x_1 < x_2$$

where  $x_1$  and  $x_2$  are any numbers in the interval.

If a function is either increasing on an interval or decreasing on an interval, then it is *monotonic* on the interval.

**Theorem 1** Let the function  $f$  be continuous on the closed interval  $[a, b]$ , and differentiable on the open interval  $(a, b)$ .

- if  $f'(x) > 0$  for all  $x \in (a, b)$ , then  $f$  is increasing on  $[a, b]$ ; else
- if  $f'(x) < 0$  for all  $x \in (a, b)$ , then  $f$  is decreasing on  $[a, b]$

**Proof:** Let  $x_1$  and  $x_2$  be any numbers in  $[a, b]$  such that  $x_1 < x_2$ .

Case 1: Suppose  $f'(x) > 0$  for all  $x \in (a, b)$ . From the hypotheses given, there exists a number  $c \in (a, b)$  such that  $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ . Since  $x_1 < x_2$ , then this means  $f(x_2) - f(x_1) > 0$  or  $f(x_1) < f(x_2)$ .

Case 2: Suppose  $f'(x) < 0$  for all  $x \in (a, b)$ . From the hypotheses given, there exists a number  $c \in (a, b)$  such that  $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ . Since  $x_1 < x_2$ , then this means  $f(x_2) - f(x_1) < 0$  or  $f(x_1) > f(x_2)$ .

*End of Proof.*

**Theorem 2 (First Derivative Test for Relative Extrema)** Let the function  $f$  be continuous at all points of the open interval  $(a, b)$  containing the number  $c$ , and suppose  $f'$  exists at all points in  $(a, b)$  except possibly at  $c$ .

- if  $f'(x) > 0$  for all values of  $x$  in some open interval having  $c$  as its right endpoint, and  $f'(x) < 0$  for all values of  $x$  in some open interval having  $c$  as its left endpoint, then  $f$  has a relative maximum value at  $c$ ; and
- if  $f'(x) < 0$  for all values of  $x$  in some open interval having  $c$  as its right endpoint, and  $f'(x) > 0$  for all values of  $x$  in some open interval having  $c$  as its left endpoint, then  $f$  has a relative minimum value at  $c$ .

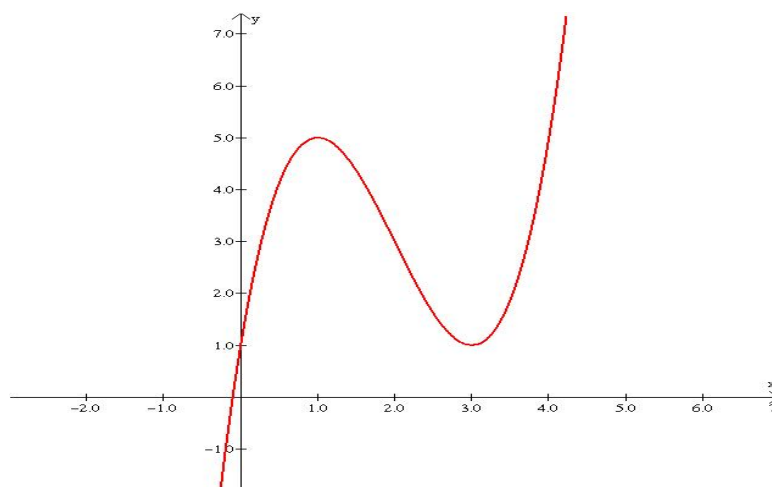
To determine analytically the relative extrema of  $f$ :

1. Get  $f'(x)$ .
2. Determine all possible critical numbers of  $f$ .
3. Apply the First Derivative Test.

In sketching the graphs, it is also good to know the roots of the function, if there are any. We now present an example.

**Example:** Find all relative extrema of the function  $f(x) = x^3 - 6x^2 + 9x + 1$  and sketch the graph. First,  $f'(x) = 3x^2 - 12x + 9$ . Equating the derivative to zero, we get critical numbers 3 and 1. We now construct the following table: The sketch of the graph is as shown below:

	$f(x)$	$f'(x)$	Conclusions
$x < 1$		+	$f$ is increasing
$x = 1$	5	0	$f$ has a relative maximum value
$1 < x < 3$		-	$f$ is decreasing
$x = 3$	1	0	$f$ has a relative minimum value
$x > 3$		+	$f$ is increasing



**Exercises:** Find all relative extrema of the function and sketch the graph.

- $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$
- $f(x) = 4x^2 + 8x + 13$
- $f(x) = x^3 - 12x + 17$
- $f(x) = 3x^4 + 4x^3 - 12x^2$