

Math 53 Lecture: The Rolle's Theorem and The Mean Value Theorem

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Lecture 8

The Mean Value Theorem is one of the most important theorems in calculus. It is used to prove most of the theorems in differential and integral calculus, as well as in numerical analysis. In order to prove the mean value theorem and appreciate its applications, we will look at its special case, the Rolle's Theorem.

Theorem 1 (Rolle's Theorem) *Let f be a function such that*

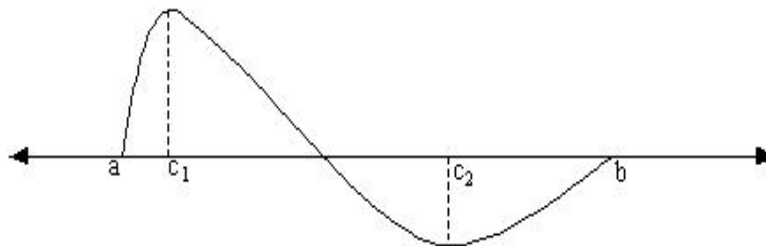
- f is continuous on $[a, b]$;
- f is differentiable in (a, b) ; and
- $f(a) = f(b) = 0$.

Then there exists a c in (a, b) such that $f'(c) = 0$.

Proof:

Case 1: Suppose $f(x) = 0$ for every $x \in [a, b]$. Then $f'(x) = 0$ for every $x \in (a, b)$. Thus every number between a and b can be c .

Case 2: Suppose $f(x) \neq 0$ for some value of $x \in (a, b)$. Since f is continuous on $[a, b]$, then by the Extreme Value Theorem, there is an absolute minimum value and an absolute maximum value on $[a, b]$. Note that $f(a) = f(b) = 0$. Then f has either a positive absolute maximum at some c_1 in (a, b) , or a negative absolute minimum at some c_2 in (a, b) , or both. In any case, there exists



an absolute extremum at an interior point of the interval $[a, b]$. Hence $f(c)$ is not only an absolute extremum value but also a relative extremum value. Since $f'(c)$ exists by hypothesis, $f'(c) = 0$.

Example: $f(x) = 4x^3 + 6x^2$; $[-\frac{3}{2}, 0]$

f is continuous in $[-\frac{3}{2}, 0]$, f is differentiable in $(-\frac{3}{2}, 0)$, and $f(-\frac{3}{2}) = f(0) = 0$. By the Rolle's Theorem, there exists a number $c \in (-\frac{3}{2}, 0)$ such that $f'(c) = 12c^2 + 12c = 0$. Solving for c and considering the interval given, $c = -1$.

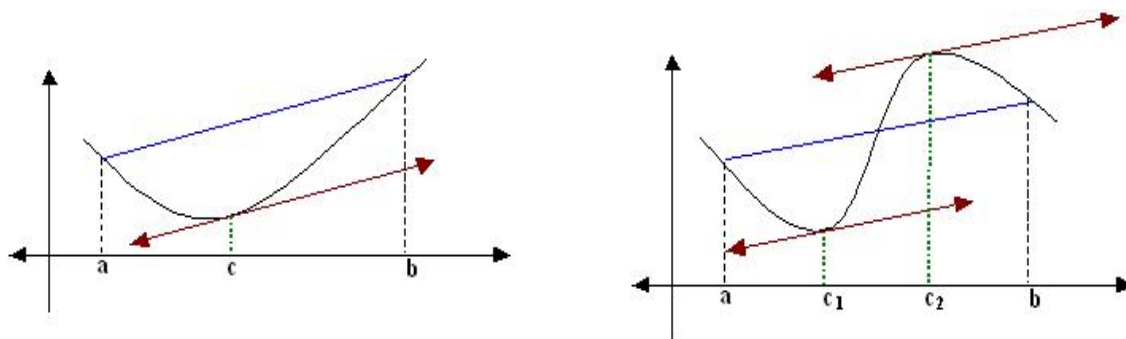
We now present the Mean Value Theorem.

Theorem 2 (Mean Value Theorem) Let f be a function such that

- f is continuous on $[a, b]$; and
- f is differentiable in (a, b) .

Then there exists a c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

Proof:



The equation of the secant line is given by $y - f(b) = m(x - b)$ where $m = \frac{f(b) - f(a)}{b - a}$. Thus the equation of the secant line may be written as

$$y = \frac{f(b) - f(a)}{b - a}(x - b) + f(b).$$

Let $F(x)$ be the measure of the vertical distance between a point $(x, f(x))$ in the graph of f and the corresponding point in the secant line. Then

$$F(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - b) - f(b).$$

Note that $F(x)$ is continuous on $[a, b]$, differentiable in (a, b) and $F(a) = F(b) = 0$. By the Rolle's Theorem, there exists a number $c \in (a, b)$ such that $F'(c) = 0$. In particular,

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \quad \text{or} \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example: $y = x^3 - x^2 - 2x$; $[1, 3]$

$f'(c) = 3c^2 - 2c - 2 = \frac{f(3) - f(1)}{3 - 1} = 7$. This gives us $3c^2 - 2c - 9 = 0$. Solving for c and considering the interval given, $c = \frac{1 + 2\sqrt{7}}{3}$.

Theorem 3 If f is a function such that $f'(x) = 0$ for all x in some interval I , then f is constant on I .

Proof: Suppose f is not constant on I . Then there exists $x_1, x_2 \in I$ such that $f(x_1) \neq f(x_2)$. Since $f'(x) = 0 \forall x \in I$ then $f'(x) = 0$ on $[x_1, x_2]$. Thus f is also continuous on $[x_1, x_2]$. By the Mean Value Theorem, there is a $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$. But $c \in I$ which means $f'(c) = 0$ so this would mean $f(x_2) = f(x_1)$ which is a contradiction.