# Math 53 Lecture: Applications Involving Absolute Extrema on a Closed Interval

Lecturer: Jose Maria L. Esaner IV, Ph.D.
Lecture 7

Recall that if a function is continuous on a closed interval then the function has absolute extremum (both minimum and maximum) values in the closed interval. Recall also the steps in finding the absolute extrema:

1. Get $f'(x)$.
2. Find all possible critical numbers $c_1, c_2, \ldots, c_n$ of $f$ on $[a, b]$ and get their functional values $f(c_i)$ for every $i = 1, 2, \ldots, n$.
3. Find $f(a)$ and $f(b)$.
4. The largest of the values in (2) and (3) is the absolute maximum value and the least is the absolute minimum value.

In working with optimization problems, two things should be given. One is the objective function, and the other is the constraint statement. The objective function, preferably expressed in a single variable, is the function that needs to be maximized or minimized. The constraint statement, on the other hand, provides the necessary condition to be included in the objective function. This also includes the interval of the independent variable in the objective function.

**Example:** Find the number in $[0, 1]$ such that the difference between the number and its square is a maximum.
If $x$ is the number in $[0, 1]$, then the difference is $x - x^2$. Hence the objective function with its constraint is

$$\max f(x) = x - x^2 \qquad x \in [0, 1].$$

The critical numbers of $f$ is $\frac{1}{2}$ so $f(\frac{1}{2}) = \frac{1}{4}$. On the other hand $f(0) = 0$ and $f(1) = 0$. Therefore $f$ has its absolute maximum value at $x = \frac{1}{2}$ and so $\frac{1}{2}$ is our desired number.

**Example:** Find the dimensions of the largest rectangular garden that can be fenced off with 100 feet of fencing material.
Our objective is to maximize the area of the rectangular region given that the perimeter is limited to 100 feet. If $l$ is the length and $w$ is the width, then the area is $A = lw$ while the perimeter is $P = 100 = 2l + 2w$. Since we need our objective function to be expressed in one variable we see that $l = 50 - w$ and so

$$\max A(w) = (50 - x)w = 50w - w^2 \qquad w \in [0, 50].$$

The critical number of $A(w)$ is $w = 25$. $A(0) = A(50) = 0$ while $A(25) = 625$ Thus the dimensions of the largest rectangular garden that can be fenced off with 100 feet of fencing material is 25 feet by 25 feet.

**Exercises:**

1. The sum of one number and three times the second number is 48. What numbers should be selected so that their product is as large as possible? ($x = 8, y = 24$)
2. A farmer has 800 m of fencing material to enclose a rectangular pen adjacent to a long existing wall. He will use the wall for one side of the pen and the available fencing material for the remaining three sides. What is the maximum area that can be enclosed this way? ($80,000m^2$)
3. A rectangular with a fixed perimeter of 64 units is rotated about one of its sides, thus sweeping out a figure in the shape of a right circular cylinder. What is the maximum volum of that cylinder? ($131,072\pi/27$)
4. A private club charges annual membership dues of Php 100 per member, less Php 0.50 for each member over 600 and plus Php 0.50 for each member less than 600. How many members will give the club the most revenue of annual dues? ($x = 400$)
5. What is the area of the largest rectangle that can be drawn in a circle of radius 8?
6. What is the area of the largest rectangle that can be drawn in a semicircle of radius $a$?
7. A 48-cm piece of string is cut into two pieces. One piece is used to form a circle and the other to form a square. How should the string be cut so that the sum of the areas is a minimum?