Math 53 Lecture: Relative Extrema and Absolute Extrema

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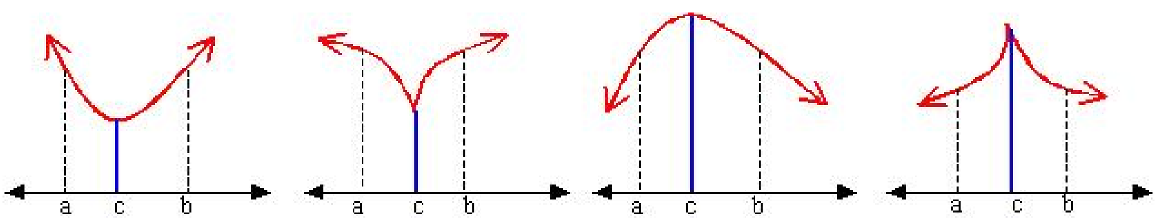
Lecture 6

Definition 1 *A function f is said to have a relative maximum (minimum) value at c if there exists an open interval containing c on which f is defined, such that $f(c) \geq f(x)$ ($f(c) \leq f(x)$) for all x in the interval.*



Theorem 1 *If $f(x)$ exists for $x \in (a, b)$ and f has a relative extremium value at c, $a < c < b$ and $f'(c)$ exists, then $f'(c) = 0$.*

The converse of the above theorem is not true as seen in the following example.

Example: The function $f(x) = (x-1)^3$ does not have relative extremum at $x = 1$ although $f'(1) = 0$.

Example: The function $f(x) = \begin{cases} x-1 & x \leq 4 \\ 7-x & x > 4 \end{cases}$ has a relative extremum at $x = 4$ although $f'(4)$ does not exist.

From the above examples, a necessary condition for f to have a relative extremum value at c is either $f'(c) =$ or $f'(c)$ does not exist. But they are not sufficient conditions.

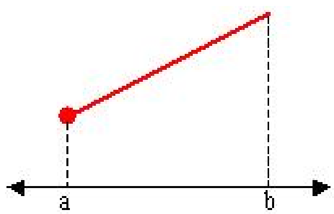
Definition 2 *If c is a number in the domain of f and $f'(c) = 0$ or $f'(c)$ does not exist, then c is called a critical number of f.*

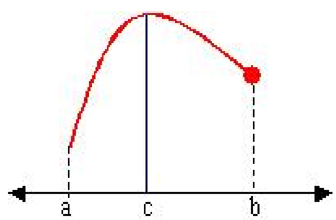
Exercises: Find all possible critical numbers of the following functions.

1. $f(x) = x^3 + 7x^2 - 5x$
2. $f(x) = 2x^3 - 2x^2 - 16x + 1$
3. $f(x) = \dfrac{x+1}{x^2 - 5x + 4}$

As a final note, relative extremum values are not necessarily relative to any specified interval. If a particular interval, be it open, closed, half-open or half-closed, then what we have is an absolute extremum value.

Definition 3 *The function f is said to have an absolute maximum (minimum) value on an interval I if there is a number $c \in I$ such that $f(c) \geq f(x)$ $(f(c) \leq f(x))$ for all x in the interval. We say $f(c)$ is the absolute maximum (minimum) value of f in I.*





Theorem 2 (The Extreme Value Theorem) *If the function f is continuous on the closed interval $[a, b]$, then f has an absolute minimum value and an absolute maximum value on $[a, b]$.*

As an application of the Extreme Value Theorem, we consider the following steps in finding the absolute minimum or absolute maximum of a function $f(x)$ on a closed interval $[a, b]$.

1. Get $f'(x)$.
2. Find all possible critical numbers $c_1, c_2, \ldots, c_n$ of f on $[a, b]$ and get their functional values $f(c_i)$ for every $i = 1, 2, \ldots, n$.
3. Find $f(a)$ and $f(b)$.
4. The largest of the values in (2) and (3) is the absolute maximum value and the least is the absolute minimum value.

Exercises: Find the absolute maximum and the absolute minimum values of the following functions given the specified closed interval.

1. $g(x) = x^3 + 5x - 4$; $[-3, -1]$
2. $f(x) = x^4 - 8x^2 + 16$; $[-4, 0]$
3. $f(t) = 2 \sin t$; $[-\pi, pi]$
4. $f(x) = \dfrac{x}{x+2}$; $[-1, 2]$
5. $F(x) = \begin{cases} 3x - 4 & \text{if } -3 \leq x < 1 \\ x^2 - 2 & \text{if } 1 \leq x \leq 3 \end{cases}$; $[-3, 3]$