Math 53 Lecture: Linear Approximations and Differentials

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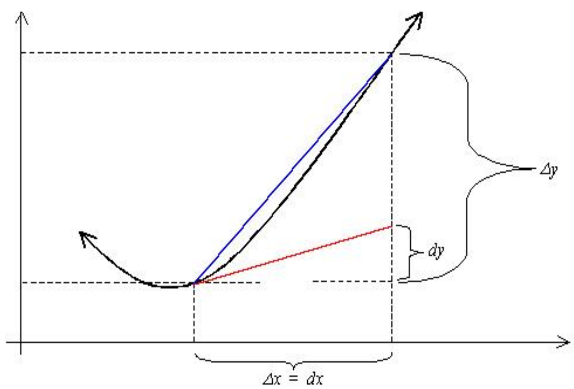
Lecture 5

Another application of the derivatives is linear approximation of a quantity. Suppose we would like to approximate the value of $\sqrt{4.01}$. How do we use derivatives to approximate this?

Let $y = f(x)$ be a differentiable function and consider the figure below:



In the above figure, Δx is called the increment of x. The increment of y is defined as

$$\Delta y = f(x + \Delta x) - f(x).$$

On the other hand, the derivative of f is defined by $f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$. For sufficiently small $|\Delta x|$, we can make a good approximation $f'(x) \approx \dfrac{\Delta y}{\Delta x}$ or equivalently, $\Delta y \approx f'(x)\Delta x$. We now define the following:

Definition 1 *If the function f is defined by the equation $y = f(x)$, then the differential of y, denoted by dy is given by $dy = f'(x)\Delta x$, and the differential of x, denoted by dx is given by $dx = \Delta x$, where x is in the domain of f' and Δx is an arbitrary increment of x.*

From the definition and the previous discussion, we have the following:

$$\begin{aligned} dy &= f'(x)dx \quad \text{and} \\ f(x+\Delta x) &\approx f(x) + f'(x)\Delta x. \end{aligned}$$

The last approximation is what will be used in computing for the $\sqrt{4.01}$. For this case, let $f(x) = \sqrt{x}$, $x_0 = 4$ and $\Delta x = .01$. Then $f'(x) = \dfrac{1}{2\sqrt{x}}$ and

$$\sqrt{4.01} = f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x = 2 + \frac{1}{4}(.01) = 2.0025.$$