Math 53 Lecture: Rectilinear Motion and Instantaneous Rates of Change

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Lecture 2

Recall that the definition of the derivative of a function at a point evolved from the slope a tangent line on a curve at a point, which is the limiting case of slope of a secant line on a curve. In a sense, what has become a usual rate of change of functional values became the *instantaneous* rate of change of the function at a number. We shall adapt this technique (in a different jargon) to give us a more practical application of the concept of derivatives.

We consider now a simple application in physics: rectilinear motion. That is, the motion of a particle on a (horizontal) line. Let us assume that the right direction is considered as the postive direction while the other direction gives a negative direction. Thus if a particle from an arbitrary point of origin moves to the right direction, the particle traverses a positive distance s. If t is the time (in seconds) then the **displacement** function from the origin at a particular instant t is given by $s = s(t)$. That is, the directed distance depends on the time the particle travels from the origin to a point on the line.

Now, suppose a particle from the origin moves at a distance to the right. Then after t_0 seconds, it has traversed $s(t_0)$ units of distance. From this point, the particle moves on either direction for h more seconds. Then *averate rate of change* in the distance, or better known as the average velocity, between time $[t_0, t_0 + h]$ is given by

$$\text{velocity} = \frac{s(t_0 + h) - s(t_0)}{(t_0 + h) - t_0} = \frac{s(t_0 + h) - s(t_0)}{h}.$$

If we now let h approach zero, then we get the **instantaneous** velocity of the particle, v, at time t_0 by

$$v(t_0) = s'(t_0) = \frac{d}{dt}s(t_0) = \lim_{h \to 0} \frac{s(t_0 + h) - s(t_0)}{h}.$$

In general, the instantaneous velocity of the displacement function at any time t is the first derivative with respect to t of the displacement function, i.e., $v(t) = s'(t)$.

The instantaneous velocity may either be positive or negative depending on how the particle moved on the horizonal line. Note that when the instantaneous velocity is zero, the particle is at rest. One important term also included in rectilinear motion is speed. **Speed** is defined as the absolute value of the instantaneous velocity. Thus the instantaneous velocity is a directional (vector) parameter while the speed is a scalar parameter.

The second derivative of the displacement function, $a(t) = s''(t)$ is called the instantaneous **acceleration** of the particle. It is so termed since it is the instantaneous rate of change in the velocity of the particle.

The third derivative of the displacement function, $j(t) = s'''(t)$ is called the (instantaneous) **jerk** - the instantaneous rate of change in the acceleration of the particle.

Exercises: A particle is moving along a horizontal line according to the given equation $s(t)$ where s is measured in meters and t is measured in seconds. Find, if they exist, the displacement $s(t)$, the intantaneous velocity $v(t)$, instantaneous acceleration $a(t)$ and the instantaneous jerk $j(t)$ at the given particular t_1.

1. $s(t) = 4t^3 - 3t^2 + 5t - 1$, $t_1 = 0$
2. $s(t) = \dfrac{2}{t-1}$, $t_1 = 2$
3. $s(t) = -16t^2 + 32t + 12$, $t_1 = 4$
4. $s(t) = cos3t - sin2t$, $t_1 = 2$

Exercises: Solve the following problems.

1. Free-falling bodies: A ball is dropped from a height of 30 feet from the horizontal ground. The height h in feet at any time t seconds is given by $h(t) = -16t^2 + 30$. Find
   (a) the average velocity from $t = 1$ to $t = 2$;
   (b) the instantaneous velocity at $t = 1$ and at $t = 2$;
   (c) the time when ball reaches the ground; and
   (d) the the velocity of the ball as it reaches the ground.
2. Curvilinear motion: A ball is thrown vertically upward from the ground with an initial velocity of 20 feet per second. If the positive direction of the distance from the starting point is up, the equation of the motion is given by $s(t) = -16t^2 + 20t$.
   (a) the instantaneous velocity at $t = 3$;
   (b) the time for the ball to reach the highest point;
   (c) the total time for the ball to reach the ground; and
   (d) the instantaneous velocity as the ball reaches the ground.
3. Curvilinear motion: Carlo throws a ball upward from a window in a building 20 feet above the ground. If the initial velocity is 13 feet per second, the equation of the motion is given by $s(t) = -16t^2 + 13t + 20$. Find
   (a) the instantaneous velocity at $t = 1$;
   (b) the time for the ball to reach the highest point;
   (c) the total time for the ball to reach the ground; and
   (d) the instantaneous velocity as the ball reaches the ground.