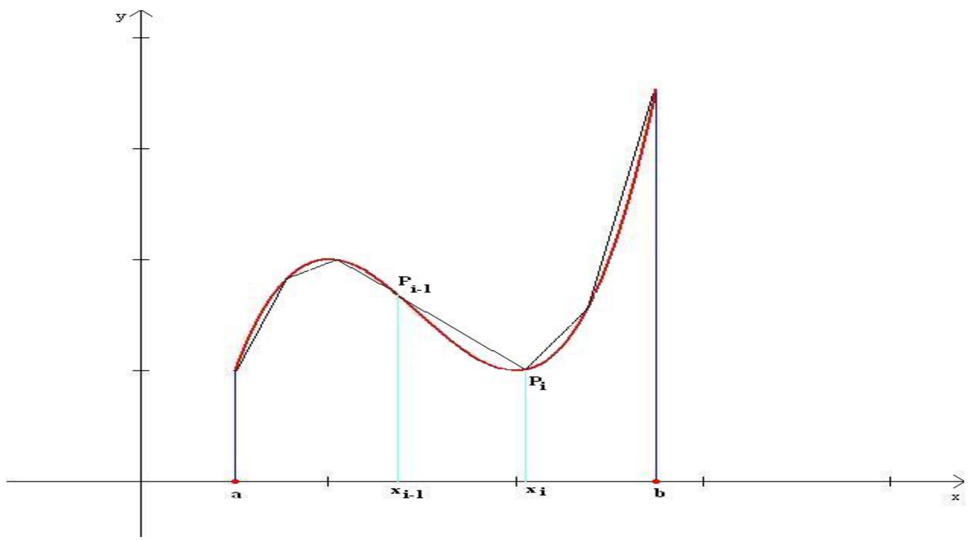
Math 53 Lecture: Length of an Arc

Lecturer: Jose Maria L. Esaner IV, Ph.D.
Lecture 21

Another application of the definite integral is finding the length of a smooth arc defined by a function $y = f(x)$ from $x = a$ to $x = b$. A smooth arc on $[a, b]$ is a graph of a smooth function f on $[a, b]$ such that its derivative f' is continuous on $[a, b]$. Loosely speaking, smooth arcs rules out any "sharp" corners on the graph of f. Consider the diagram below:



In order to get the length of this arc, we partition the curve by subintervals $x_0 = a$, $x_n = b$ and $\Delta_i x = x_i - x_{i-1}$. In the figure, the i^{th}-subinterval has endpoints P_{i-1} and P_i with coordinates $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$, respectively. The distance between these two points is given by $\Delta_i s = |P_{i-1}P_i| = \sqrt{[x_i - x_{i-1}]^2 + [f(x_i) - f(x_{i-1})]^2}$. So an approximation to the length s of the arc is

$$s \approx \sum_{i=1}^{n} |P_{i-1}P_i|.$$

We apply the mean value theorem to the function f on the subinterval $[x_{i-1}, x_i]$. That is, there exists a number $\xi_1 \in [x_{i-1}, x_i]$ such that $[f(x_i) - f(x_{i-1})] = f'(\xi_i)[x_i - x_{i-1}]$.

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{[x_i - x_{i-1}]^2 + [f(x_i) - f(x_{i-1})]^2} \\ &= \sqrt{[x_i - x_{i-1}]^2 + f'(\xi_i)[x_i - x_{i-1}]^2} \\ &= \sqrt{(\Delta_i x)^2 + (f'(\xi_i)\Delta_i x)^2} \\ &= \sqrt{1 + [f'(\xi_i)]^2}\Delta_i x \end{aligned}$$

We now take the limit of the Riemann sum of all $|P_{i-1}P_i|$ to arrive at the actual length of the arc:

$$s = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(\xi_i)]^2}\Delta_i x = \int_a^b \sqrt{1 + [f(x)]^2}\ dx.$$