Math 53 Lecture: Volumes of Solids Using Disk Method and Washer Method

Lecturer: Jose Maria L. Esaner IV, Ph.D.
Lecture 19

Consider a plane region R formed by the curve $y = f(x)$ above the x-axis, the $x - axis$, and the lines $x = a$ and $x = b$, $a < b$. Suppose we rotate this region about the x-axis. Then the result is a *solid of revolution*. The problem now is determine the volume of this solid of revolution.

In order to do this, we partition the region with rectangular elements parallel to the y-axis (or equivalently, rectangular elements perpendicular to the x-axis). Rotating a rectangular element, as well, produces a disk with radius $f(\psi_i)$, where $\xi_i \in [x_{i-1}, x_i]$, and height $\Delta_i x$. Recall that the volume of disk with these dimensions is given by $\Delta_i V = \pi[f(\xi_i)]^2 \Delta_i x$. Using the same argument of getting the limit of the Riemann sum, we get

$$V = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} \pi[f(\xi_i)]^2 \Delta_i x = \pi \int_a^b [f(x)]^2 \; dx.$$

In general, consider a plane region bounded by curves $f(x)$ and $g(x)$, and the lines $x = a$ and $x = b$. Without loss of generality, assume that $f(x) > 0$ and $g(x) > 0$, and furthermore, $f(x) \geq g(x)$ for every $x \in [a, b]$. Rotating the region will give a solid of revolution with a hollow part in the middle. If we partition the region with rectangular elements parallel to the y-axis and rotate it about the x-axis, we get a washer. The volume of this washer is then given by $\Delta_i V = \pi[f(\xi_i)^2 - g(\xi_i)^2]\Delta_i x$. Getting the limit of the Reimann sum of these rectangular elements we get

$$V = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} \pi[f(\xi_i)^2 - g(\xi)^2]\Delta_i x = \pi \int_a^b [f(x)]^2 - [g(x)]^2 \; dx.$$

Exercises: Find the volume of the solid of revolution that is generated by rotating the plane region R about the indicated axis or line.

1. $R : y = x^2, y = 0, x = 1$;   the x-axis
2. $R : y = x^2, y = 4, x = 0$ (first quadrant only);   the y-axis
3. $R : y = 1/x, y = 0, x = 0.1, x = 1$;   the x-axis
4. $R : y = x^2, x = y^2$;   the x-axis
5. $R : y = x^2, y = 4x$;   the line $x = 5$
6. $R : y = 6 - x^2, y = 2$;   the x-axis
7. $R : y = x^2, x = y^2$;   the line $y = -2$
8. $R : y = x^2, y = 8 - x^2$;   the line $x = 4$