Math 53 Lecture: Theorems for Definite Integrals

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Lecture 17

We present properties of the definite integrals that contributed in the foundation of integral calculus. Evaluation of definite integrals follow from the second fundamental theorem of calculus.

Theorem 1 *If the function f and g are integrable on the closed interval $[a, b]$ and if $f(x) \geq g(x)$ for all x in $[a, b]$, then*

$$\int_a^b f(x)\ dx \geq \int_a^b g(x)\ dx.$$

Theorem 2 *Suppose that the function f is continuous on the closed interval $[a, b]$. If m and M are the absolute minimum and absolute maximum values, respectively, of f on $[a, b]$ so that $m \leq f(x) \leq M$ for $a \leq x \leq b$, then*

$$m(b-a) \leq \int_a^b f(x)\ dx \leq M(b-a).$$

Theorem 3 (Mean Value Theorem for Integrals) *If the function is continuous on the closed interval $[a, b]$, there exists a number $c \in [a, b]$ such that*

$$\int_a^b f(x)\ dx = f(c)(b-a).$$

Definition 1 (Average Value of a Function) *If the function f is integrable on the closed interval $[a, b]$, then the* **average value** *of f on $[a, b]$ is*

$$\frac{\int_a^b f(x)\ dx}{b-a}.$$

Theorem 4 (First Fundamental Theorem of Calculus) *Let f be a continuous function on $[a, b]$ and let x be any number in $[a, b]$. If F is a function defined by*

$$F(x) = \int_a^x f(t)\ dt$$

then

$$F'(x) = f(x) \Longleftrightarrow \frac{d}{dx}\int_a^x f(t)\ dt = f(x).$$

Theorem 5 (Second Fundamental Theorem of Calculus) *Let f be a continuous function on $[a, b]$ and let F be a function such that*

$$F'(x) = f(x) \qquad \forall x \in [a, b].$$

Then

$$\int_a^b f(t)\ dt = F(b) - F(a).$$

Exercises:

1. Perform the following:
   (a) $\displaystyle\int \left(\sqrt[4]{x+1} - \frac{x^2+1}{\sqrt{x}} \right)\, dx$
   (b) $\displaystyle\int \sec^3 x \ \sin 2x \ dx$
   (c) $\displaystyle\int \frac{x^2}{\sqrt{1-x}}\, dx$
   (d) $\displaystyle\int \frac{1-2\sqrt{x}}{\sqrt{x}(3+2\sqrt{x})^3}\, dx$
2. Evaluate the following definite integrals:
   (a) $\displaystyle\int_0^1 (3x^2 + 2\sqrt{x} + 3\sqrt[3]{x})\, dx$
   (b) $\displaystyle\int_{-1}^0 (x+1)^3\, dx$
   (c) $\displaystyle\int_{-1}^2 (3x^2 + 2x + 4)\, dx$
   (d) $\displaystyle\int_1^4 \frac{x^2-1}{\sqrt{x}}\, dx$
   (e) $\displaystyle\int_0^\pi \sin^2 x \cos x \ dx$
   (f) * $\displaystyle\int_{-2}^2 |1-x|\, dx$
   (g) * $\displaystyle\int_{-3}^3 |3x-2|\, dx$
   (h) * $\displaystyle\int_0^6 \sqrt{6x-x^2}\, dx$
3. Use properties of integrals to establish each inequality without evaluating the integrals involved:
   (a) $1 \le \displaystyle\int_0^1 \sqrt{1+x^2}\, dx \le \int_0^1 \sqrt{1+x}\, dx$
   (b) $\displaystyle\int_1^2 \sqrt{1+x}\, dx \le \int_1^2 \sqrt{1+x^3}\, dx \le \sqrt{10}$
   (c) $\displaystyle\int_0^2 \sin\sqrt{x}\, dx \le 2$
4. Given an upper bound and a lower bound for the following definite integrals:
   (a) $\displaystyle\int_0^1 \frac{1}{1+x}\, dx$
   (b) $\displaystyle\int_4^9 \frac{1}{1+\sqrt{x}}\, dx$
   (c) $\displaystyle\int_0^{\pi/6} \cos^2 x\, dx$
   (d) $\displaystyle\int_0^{\pi/4} \sqrt{16+2\sin^2 x}\, dx$