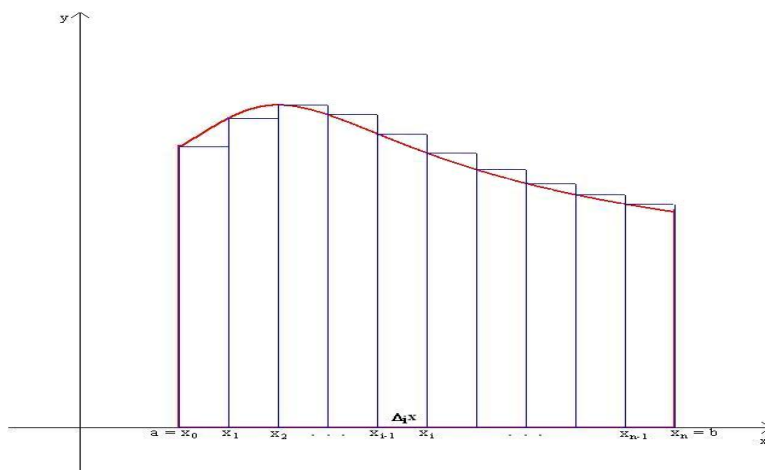


# Math 53 Lecture: Definite Integral

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## Lecture 16

We now make an abstraction of what we have done previously. Let  $R$  be a region bounded by the function  $y = f(x)$ , the  $x$ -axis and the lines  $x = a$  and  $x = b$ ,  $a < b$ . Let  $\Delta$  be a partition derived by subdividing the interval  $[a, b]$  into  $n$  subintervals, not necessarily of equal sizes, such that  $x_i = x_{i-1} + \Delta_i x$  for  $i = 1, \dots, n$ . Let  $\|\Delta\|$  be the norm of the partition, that is, the largest width among the subintervals formed. Without loss of generality, we note that the height of each rectangle



is determined by  $f(x_{i-1})$ ,  $i = 1, \dots, n$ . The area of each rectangle is given by

$$\Delta A_i = \text{length} \times \text{width} = f(x_{i-1})\Delta_i x \quad i = 1, 2, \dots, n.$$

Summing up the areas, we get what is called the *Riemann sum*

$$\sum_{i=1}^n \Delta A_i = \sum_{i=1}^n f(x_{i-1})\Delta_i x.$$

As previously mentioned, the true area of the region is achieved by letting the  $n \rightarrow \infty$ , or equivalently, letting  $\|\Delta\| \rightarrow 0$ :

$$A = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Delta A_i = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_{i-1})\Delta_i x = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_{i-1})\Delta_i x.$$

The rightmost side of the equality is called the limit of the Riemann sum and is more denoted by the simple  $\int$ .

$$A = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(x_{i-1})\Delta_i x = \int_a^b f(x) dx$$

to be read as “the definite integral of  $f$  with respect to  $x$  from  $a$  to  $b$ .”

**Definition 1** Let  $f$  be a function whose domain includes the closed interval  $[a, b]$ . Then  $f$  is said to be integrable on  $[a, b]$  if there is a number  $L$  such that for any  $w_i$  in a subinterval  $[x_{i-1}, x_i]$ ,  $x_0 = a, x_n = b$ , and for every partition  $\Delta$  for which its norm is sufficiently small, then

$$\lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(w_i) \Delta_i x = L.$$

**Definition 2** If  $f$  is a function defined on the closed interval  $[a, b]$ , then the definite integral of  $f$  from  $a$  to  $b$ , denoted by  $\int_a^b f(x) dx$ , is given by

$$\int_a^b f(x) dx = \lim_{\|\Delta\| \rightarrow 0} \sum_{i=1}^n f(w_i) \Delta_i x$$

provided the limit exists.

**Theorem 1** If a function is continuous on the closed interval  $[a, b]$ , then it is integrable on  $[a, b]$ .

**Theorem 2** We have the following properties for definite integrals:

1. If  $a > b$ , then  $\int_a^b f(x) dx = - \int_b^a f(x) dx$ .
2. If  $f(a)$  exists, then  $\int_a^a f(x) dx = 0$ .
3. If  $f$  is integrable on  $[a, b]$  and if  $k$  is any constant, then  $\int_a^b kf(x) dx = k \int_a^b f(x) dx$ .
4. If  $f$  and  $g$  are integrable on  $[a, b]$ , then  $f \pm g$  is integrable on  $[a, b]$  and  $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$ .
5. If  $f$  is integrable on a closed interval containing three numbers  $a, b$ , and  $c$ , then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

regardless of the order of the numbers.