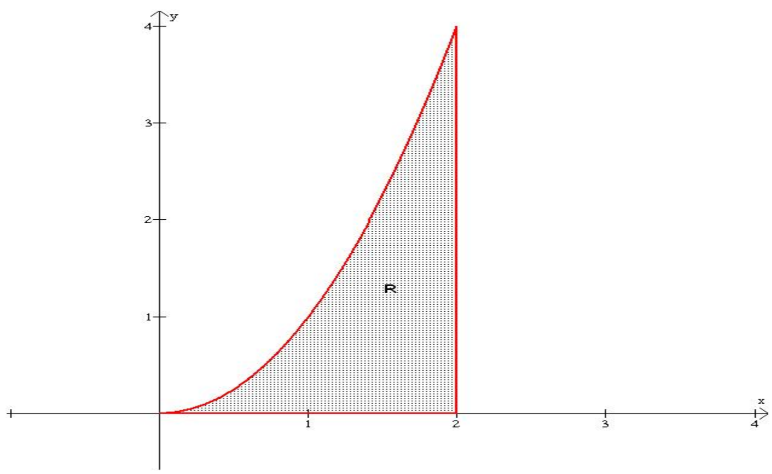
Math 53 Lecture: The Area Under the Curve

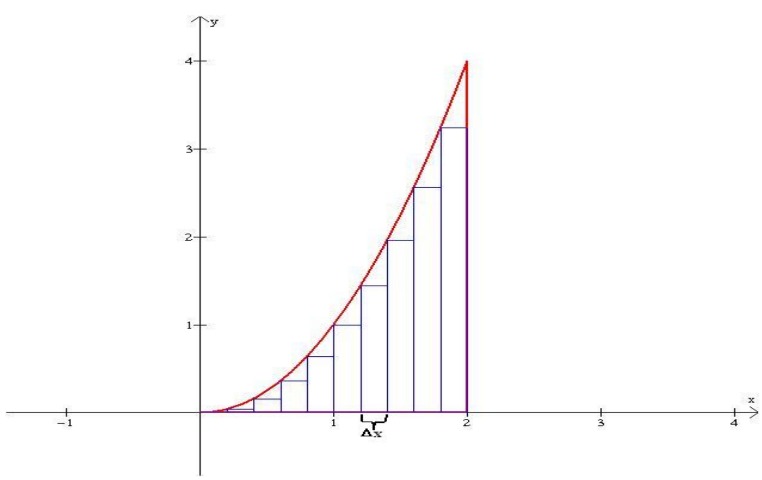
Lecturer: Jose Maria L. Esaner IV, Ph.D.
Lecture 15

In Euclidean geometry, one is familiar in the formulas for the areas of different geometric figures such as the triangle, rectangle, square, trapezoids and circles, to name a few. In this discussion, we now present ways of getting the region bounded by a curve. Ultimately, we intend to present the concept of *integral calculus*.

Suppose we consider the region bounded (enclosed) by the coordinate axes, the curve $y = x^2$ and the line $x = 2$ as shown below: We now approximate the area by considering inscribed rectangles



in the region as shown: In this case, we subdivided the interval [0,2] into 10 equally spaced strips



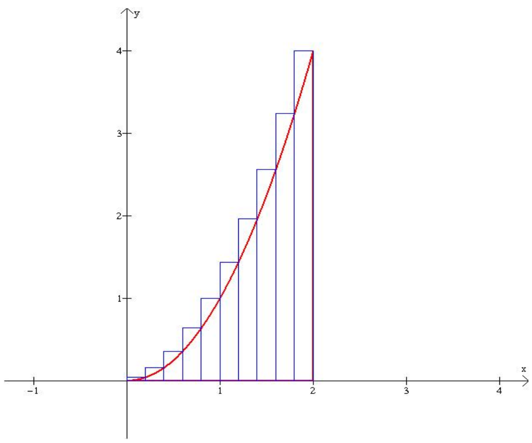
of inscribed rectangles. The width is therefore a constant of size 0.2 units. The height is actually calculated from the lower end of the each subinterval. Below is the table of values for the height.

Table 1: Values of $y = x^2$ using the left endpoints of the subintervals.

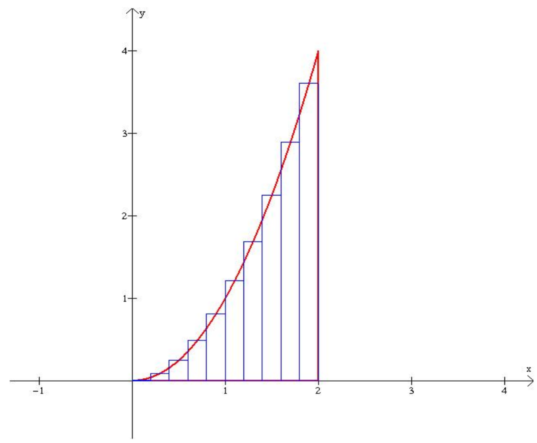
x	0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8
0	0.00	0.04	0.16	0.36	0.64	1.00	1.44	1.96	2.56	3.24

An approximate area of the region is the sum of areas of the inscribed rectangles, that is, 11.4(0.2)=2.28 square units. Note that this is an underestimation of the true area of the region.

An alternative is by using circumscribed rectangles,in which case we have an overestimation of the true area of the region. Or, we can also consider rectangles with their heights computed from the midpoints of each subinterval. In any case, the sum of the area of the rectangles is an overestimation

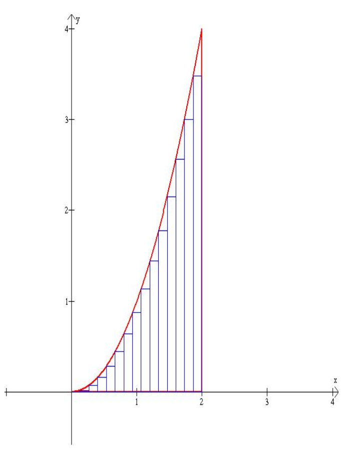


Circumscribed Rectangles, Area = 3.08

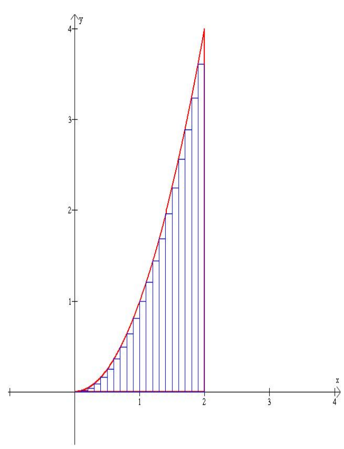


Midpoint Rectangles, Area = 2.66

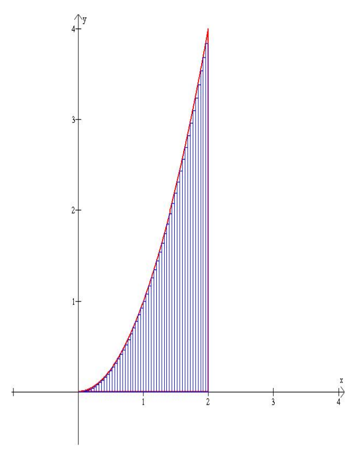
or/and underestimation of the true area of the region. To minimize the error in the approximation, the number of rectangles should be increased (or equivalently, we should increase the number of subintervals, making the width of each subinterval approach zero). Even as we increase the number



$n = 15; A = 2.40593$



$n = 20; A = 2.47$



$n = 50; A = 2.5872$

of subintervals, the error, although very small, is still there. Thus it would require us to increase the number of subintervals to increase without bound to get the exact area of the region.

Without of loss of generality, we make n subdivisions of equal length in the interval [0,2]. Define $\Delta x = \dfrac{2-0}{n}$. Observe that $f(x_0) = 0$, $f(x_1) = f(x_0 + \Delta x) = f\left(\dfrac{2}{n}\right) = \dfrac{4}{n^2}$, $f(x_2) = f\left(\dfrac{4}{n}\right) = \dfrac{16}{n^2}$ and so on. Therefore for $i = 1, \ldots, n$, $f(x_i) = f\left(\dfrac{2i}{n}\right) = \dfrac{4i^2}{n^2}$. Therefore, the area of the ith subinterval is given by

$$\Delta_i A = f(x_i)\Delta x = \frac{4i^2}{n^2}\,\frac{2}{n} = \frac{8i^2}{n^3}.$$

Getting the sum of all the rectangles,

$$\sum_{i=1}^{n} \Delta_i A = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \frac{8i^2}{n^3}.$$

Since we would like to get the real area of the region, we now let n approach infinity.

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i A \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{8i^2}{n^3} \\
&= \lim_{n\to\infty} \frac{8}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} \frac{4(2n^3+3n^2+n)}{3n^3} \\
A &= \frac{8}{3} = 2.666\overline{6}.
\end{aligned}$$

Appendix: The Sigma Notation

Definition 1 *For any integers m and n, $m \leq n$, we have*

$$\sum_{i=m}^{n} F(i) = F(m) + F(m+1) + F(m+2) + \cdots + F(n-1) + F(n).$$

Theorem 1 $\displaystyle\sum_{i=1}^{n} c = cn$ *where c is any constant.*

Theorem 2 $\displaystyle\sum_{i=1}^{n} cF(i) = c\sum_{i=1}^{n} F(i)$ *where c is a constant.*

Theorem 3 $\displaystyle\sum_{i=1}^{n}[F(i) \pm G(i)] = \sum_{i=1}^{n} F(i) \pm \sum_{i=1}^{n} G(i).$

Theorem 4 $\displaystyle\sum_{i=a}^{b} F(i) = \sum_{i=a+c}^{b+c} F(i-c).$

Theorem 5 $\displaystyle\sum_{i=a}^{b} F(i) = \sum_{i=a-c}^{b-c} F(i+c).$

Theorem 6 (Telescopic Sum) $\displaystyle\sum_{i=1}^{n}[F(i) - F(i-1)] = F(n) - F(0).$

Theorem 7 *If n is a positive integer, then*

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2}. \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6}. \\
\sum_{i=1}^{n} i^3 &= \frac{n^2(n+1)^2}{4}. \\
\sum_{i=1}^{n} i^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.
\end{aligned}$$