Math 53 Lecture: Techniques of Antidifferentiation

Lecturer: Jose Maria L. Esaner IV, Ph.D.
Lecture 13

In our previous discussion, we were able to get antiderivatives of certain familiar functions. We now present some techniques that requires the idea of chain rule and that of changing variables.

Suppose we are interested in finding $\int 6(2x+1)^5\ dx$. One way to do this is by expanding the binomial then do the antidifferentiation.

$$\begin{aligned}\int 6(2x+1)^5\ dx &= \int 6(32x^5+80x^4+80x^3+40x^2+10x+1)\ dx \\ &= \int 192x^5+480x^4+480x^3+240x^2+60x+6\ dx \\ &= 32x^6+96x^5+120x^4+80x^3+120x^2+6x+C.\end{aligned}$$

This might still be easy for some (and algebraically challenging for the others!). But what if we want to get $\int (x+3)^{100}\ dx$? Obviously this will be very taxing to anyone. Is there an easier way to do the actual antidifferentiation? The answer is of course, yes. In order to do this, let us recall the chain rule for differentiation.

Theorem 1 *If $g(x)$ is a differentiable function of x, and $f(x) = F'(x)$ then*

$$\frac{d}{dx}[F(g(x))] = f(g(x))g'(x).$$

From this theorem we argue the following: if we let $u = g(x)$ then $\dfrac{d}{dx}u = g'(x)$ or, in particular, $du = g'(x)dx$. Since $\dfrac{d}{dx}[F(g(x))] = f(g(x))g'(x) = f(g(x))\dfrac{dg(x)}{dx}$, then $dF(g(x)) = f(g(x))g'(x)dx$. We therefore have

$$dF(u) = f(u)du, \qquad \text{where } u = g(x),$$

and thus

$$\int f(u)du = F(u) + C.$$

This is the chain rule for antidifferentiation. Going back to our example $\int 6(2x+1)^5\ dx$, if we let $u = 2x+1$ then $du = 2dx$ or that $\dfrac{1}{2}du = dx$. Substituting this to our problem,

$$\begin{aligned}\int 6(2x+1)^5\ dx &= \int 6u^5\left(\frac{1}{2}\right)du \\ &= 3\int u^5\ du \\ &= \frac{1}{2}u^6 + C \\ &= \frac{1}{2}(2x+1)^6 + C.\end{aligned}$$

Exercises: Perform the following:

1. $\int (x+3)^{100}\ dx$
2. $\int (3x+2)^4\ dx$
3. $\int (x^2+4x-1)^2(2x+4)\ dx$
4. $\int \left(\frac{x^3-6x}{3}\right)(x^2-2)\ dx$
5. $\int 2\sin x\cos x\ dx$
6. $\int \cos^2 x\ dx$
7. $\int \frac{\sec\sqrt{x}\ \tan\sqrt{x}}{\sqrt{x}}\ dx$
8. $\int \frac{\sin x}{(21-2\cos x)^3}\ dx$
9. $\int \sec^2(\sec x)\sec x\tan x\ dx$
10. $\int \sqrt{3+x}(x+1)^2\ dx$
11. $\int \frac{x^3}{(x^2+4)^{3/2}}\ dx$
12. $\int \frac{x^3}{\sqrt{1-2x}}\ dx$