

# Math 53 Lecture: Concavity, Points of Inflection and the Second Derivative Test

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Lecture 10

**Definition 1** *The graph of a function  $f$  is said to be concave upward (downward) at the point  $(c, f(c))$  if  $f'(c)$  exists and if there is an open interval  $I$  containing  $c$  such that for all values of  $x \neq c$  in  $I$  the point  $(x, f(x))$  on the graph is above (below) the tangent line to the graph at  $(c, f(c))$ .*

If a function is either increasing on an interval or decreasing on an interval, then it is *monotonic* on the interval.

**Theorem 1** *Let the function  $f$  be differentiable on some open interval  $I$  containing  $c$ .*

- if  $f''(c) > 0$  then the graph of  $f$  is concave upward at  $(c, f(c))$ .
- if  $f''(c) < 0$  then the graph of  $f$  is concave downward at  $(c, f(c))$ .

The proof of this theorem entails some analytic geometry and will be left as an exercise to the student. In the theorem, the only cases considered are  $f''(c) > 0$  or  $f''(c) < 0$ . So what happens when  $f''(c) = 0$ ? If there is a point on the graph of the function at which the concavity changes and the graph has a tangent line there, then the graph crosses its tangent line at this point.

**Definition 2** *The point  $(c, f(c))$  is a point of inflection of the graph of the function  $f$  if the graph has a tangent line there, and if there exists an open interval  $I$  containing  $c$  such that if  $x$  is in  $I$ , then either*

- $f''(x) < 0$  if  $x < c$ , and  $f''(x) > 0$  if  $x > c$ ; or
- $f''(x) > 0$  if  $x < c$ , and  $f''(x) < 0$  if  $x > c$ .

**Theorem 2** *Let  $f$  be a differentiable function on some open interval containing  $c$ , and  $(c, f(c))$  be a point of inflection of the graph of  $f$ . Then if  $f''(c)$  exists, then  $f''(c) = 0$ .*

The converse of the theorem is not true. For example, if  $f(x) = x^4$ , then  $f''(0) = 0$ . But the origin is not a point of inflection. As a matter of fact the graph of  $f$  is concave upward everywhere.

**Theorem 3 (Second Derivative Test for Relative Extrema)** *Let  $c$  be a critical number of a function  $f$  at which  $f'(c) = 0$ , and let  $f''$  exist for all values of  $x$  in some open interval containing  $c$ ,*

- if  $f''(c) > 0$ , then  $f$  has a relative maximum value at  $c$ ; and
- if  $f''(c) < 0$ , then  $f$  has a relative minimum value at  $c$ .

To determine analytically the relative extrema and points of inflection, if any, of  $f$ :

1. Get  $f'(x)$  and  $f''(x)$ .
2. Determine all possible critical numbers of  $f$ .

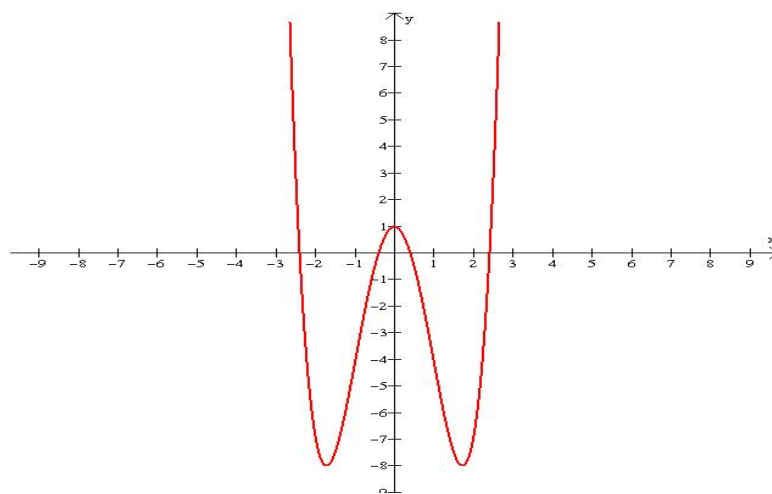
3. Equate  $f''(x)$  to zero and solve all possible points of inflection.

4. Apply the First and Second Derivative Tests.

**Example:** Find all relative extrema and points of inflection, if any, of the function  $f(x) = x^4 - 6x^2 + 1$ . Determine the intervals for which the graph is increasing/decreasing and concave upward/downward. Sketch the graph.

First,  $f'(x) = 4x^3 - 12x$ . Equating the derivative to zero, we get critical numbers  $\pm\sqrt{3}$  and 0. Now  $f''(x) = 12x^2 - 12$ . Equating this to zero, the possible point of inflection may occur when  $x = \pm 1$ . We now construct the following table: The sketch of the graph is as shown below:

	$f(x)$	$f'(x)$	$f''(x)$	Conclusions
$x < -\sqrt{3}$		-	+	$f$ is decreasing; $f$ is concave upward
$x = -\sqrt{3}$	-8	0	+	$f$ has a relative minimum value $f$ is concave upward
$-\sqrt{3} < x < -1$		+	+	$f$ is increasing $f$ is concave upward
$x = -1$	-4	+	0	$f$ is increasing $f$ has a point of inflection
$-1 < x < 0$		+	-	$f$ is increasing $f$ is concave downward
$x = 0$	0	0	-	$f$ has a relative maximum value $f$ is concave downward
$0 < x < 1$		-	-	$f$ is decreasing $f$ is concave downward
$x = 1$	-4	-	0	$f$ has a point of inflection $f$ is concave downward
$1 < x < \sqrt{3}$		-	+	$f$ is decreasing $f$ is concave upward
$x = \sqrt{3}$	-8	0	+	$f$ has a relative minimum value $f$ is concave upward
$x > \sqrt{3}$		+	+	$f$ is increasing $f$ is concave upward



**Exercises:** Find all relative extrema and point(s) of inflection of the function and sketch the graph. Make a table indicating where the function has a relative extrema, where it has an inflection, where it is decreasing/increasing, and where it is concave upward/downward.

1.  $f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + 1$

2.  $f(x) = 4x^2 + 8x + 13$

3.  $f(x) = x^3 - 12x + 17$

4.  $f(x) = 3x^4 + 4x^3 - 12x^2$