Determinants

Let $S_n = \{1, 2, \ldots, n\}$ be the set of integers from 1 to n. A rearrangement of the elements in S is called a permutation of S_n. The possible number of permutations S_n can have is $n!$, the factorial of n, which is defined by

$$n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1.$$

Consider the permuation $j_1 j_2 j_3 \ldots j_r \ldots j_s \ldots j_n$. A permutation is said to be an inversion if a larger j_r precedes a smaller j_s. A permutation is even (odd) if the total number of inversions is even (odd).

Example: $S_3 = \{(1\ 2\ 3), (1\ 3\ 2), (2\ 1\ 3), (2\ 3\ 1), (3\ 2\ 1), (3\ 1\ 2)\}$
the odd permutations are: $\{(1\ 3\ 2), (3\ 2\ 1), (2\ 1\ 3)\}$ and
the even permuations are : $\{(1\ 2\ 3), (3\ 1\ 2), (2\ 3\ 1)\}$.

Definition: Let $A = [a_{ij}]$ be an $n \times n$ matrix. The determinant of A, denoted by $|A|$, is given by

$$|A| = \sum_{p \to n!} \epsilon_{j_1 j_2 \cdots j_n}\, a_{1j_1}\, a_{2j_2} \ldots a_{nj_n}$$

where

$$\epsilon_{j_1 j_2 \cdots j_n} = \begin{cases} +1 & \text{if permutation is even} \\ -1 & \text{if permutation is odd} \end{cases}.$$

Examples:

1. If A is a 1×1 matrix; i.e. $A = [a_{11}]$, then $|A| = a_{11}$.

2. Determinant of order 2

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \epsilon_{12} a_{11} a_{22} + \epsilon_{21} a_{21} a_{12} = a_{11} a_{22} - a_{21} a_{12}.$$

3. Determinant of order 3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{aligned}[t] & \epsilon_{123}a_{11}a_{22}a_{33} + \epsilon_{231}a_{12}a_{23}a_{31} + \epsilon_{321}a_{13}a_{22}a_{31} \\ + \; & \epsilon_{213}a_{12}a_{21}a_{33} + \epsilon_{132}a_{11}a_{23}a_{32} + \epsilon_{312}a_{13}a_{21}a_{32} \end{aligned}$$

$$= a_{11}\begin{vmatrix} a_{2}2 & a_{2}3 \\ a_{32} & a33 \end{vmatrix} - a_{12}\begin{vmatrix} a_{2}1 & a_{2}3 \\ a_{31} & a33 \end{vmatrix} + a_{13}\begin{vmatrix} a_{2}1 & a_{2}2 \\ a_{31} & a32 \end{vmatrix}.$$

Exercises:

1. $\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix}$

2. $\begin{vmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 5 & 6 & 7 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 0 & 6 \\ 3 & 4 & 15 \\ 5 & 6 & 21 \end{vmatrix}$

4. $\begin{vmatrix} 2 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$

5. $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 5 \\ 4 & 1 & 3 \end{vmatrix}$

Properties of Determinants

1. $|A^T| = |A|$.
2. If matrix B results from interchanging 2 rows (or columns) of matrix A, $|B| = -|A|$.
3. If two rows (or columns) of A are equal, then $|A| = 0$.
4. If a row (or column) consists of entirely zero, then $|A| = 0$.
5. If matrix B results from multiplying a row of matrix A by a scalar c, then $|B| = c|A|$.