Math 114 Lecture: The Reduced Row Echelon Form of a Matrix

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Lecture 5

Before we go into the actual solving of linear systems, we introduce a technique of manipulating a matrix into a "reduced" form that will be used in solving linear systems.

Definition 1 *An $m \times n$ matrix is said to be in* **reduced row echelon form (rref)** *when it satisfies the following:*

1. *All zero rows (rows consisting entirely of zeroes), if any, are at the bottom of the matrix.*
2. *The first non-zero entry of each non-zero row is 1 (called the leading entry of the row).*
3. *If rows i and $i+1$ are two successive non-zero rows, then the leading entry of row $i+1$ is to the right of the leading entry of row i.*
4. *If a column contains a leading entry of some row, then all other entries in that column are zero.*

Examples:

1. $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is in reduced row echelon form.
2. $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in reduced row echelon form.
3. $C = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ is NOT in reduced row echelon form.
4. $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is NOT in reduced row echelon form.

We now discuss the tools needed in transforming a matrix in its reduced row echelon form.

Definition 2 *An* **elementary row operation** *on an $m \times n$ matrix $A = [a_{ij}]$ is any (or combination thereof) of the following operations:*

1. *Interchange rows r and s of A.*
2. *Mutliply row r of A by some constant $c \neq 0$.*
3. *Add row r of A to row s of A, $r \neq s$.*

There is one important note to be taken: when we transform a matrix in its reduced row echelon form, the transformed matrix is NOT equal to the original matrix, and therefore we should not write the "=" sign while doing the transformation. Instead, we can use the "$\Leftrightarrow$" sign.

Example: Transform the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ in its reduced row echelon form.

Solution:

$$\begin{bmatrix} 0 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\Longleftrightarrow} \begin{bmatrix} 4 & 1 & 8 \\ 2 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \overset{\frac{1}{4}R_1 \to R_1}{\Longleftrightarrow} \begin{bmatrix} 1 & \frac{1}{4} & 2 \\ 2 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \overset{2R_1 - R_2 \to R_2}{\Longleftrightarrow} \begin{bmatrix} 1 & \frac{1}{4} & 2 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\overset{\frac{2}{3}R_2 \to R_2}{\Longleftrightarrow} \begin{bmatrix} 1 & \frac{1}{4} & 2 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 2 \end{bmatrix} \overset{\frac{1}{2}R_3 \to R_3}{\Longleftrightarrow} \begin{bmatrix} 1 & \frac{1}{4} & 2 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix} \overset{R_2 - \frac{2}{3}R_3 \to R_2}{\Longleftrightarrow} \begin{bmatrix} 1 & \frac{1}{4} & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R_1 - 2R_3 \to R_1}{\Longleftrightarrow} \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{R_1 - 4R_2 \to R_1}{\Longleftrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 3 *An $m \times n$ matrix A is said to be* **row equivalent** *to an $m \times n$ matrix B if B can be obtained by applying finite sequence of elementary row operations to A.*

In our example, we say that $\begin{bmatrix} 0 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ is row equivalent to the identity matrix.

Theorem 1 *Every non-zero $m \times n$ matrix is row equivalent to a unique matrix in reduced row echelon form.*

Exercises: Transform the following matrices in its reduced row echelon form.

1. $A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 3 & -2 & 5 & 1 \\ 3 & -1 & 2 & 0 & 4 \\ 4 & -5 & 6 & -5 & 7 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{bmatrix}$

4. $A = \begin{bmatrix} 0 & 1 & 3 & -2 \\ 0 & 4 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 5 & -3 & 4 \end{bmatrix}$

Problem: Find all the possible 2 matrices which are in reduced row echelon form.