

Math 114 Lecture: Properties of Matrix Operations

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Lecture 3

In our previous discussion, we have already realized that for two square matrices A and B , matrix multiplication is not commutative. We now list other properties of matrix operations that will be useful to us. These properties are very much like properties of real numbers except that in some cases some may not apply, e.g. commutativity in multiplication. The properties are stated as theorems and proofs will be discussed in class.

Theorem 1 [*Properties of Matrix Addition*] Let A , B and C be $m \times n$ matrices. Then

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. There is a unique $m \times n$ matrix 0 , such that $A + 0 = 0 + A = A$, called the zero matrix.
4. For each matrix A , there is a unique $m \times n$ matrix G such that $A + G = 0$. We now write $G = -A$ such that $A + (-A) = 0$. In this case $-A$ is called the negative of A .

Theorem 2 [*Properties of Matrix Multiplication*] Let A , B and C be matrices of appropriate sizes. Then

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. If A is an $m \times n$ matrix then $I_m A = A I_n = A$

Theorem 3 Any $n \times n$ scalar matrix may be written as rI_n for some r .

Definition 1 If A is $n \times n$, $A^0 = I_n$.

Theorem 4 [*Properties of Powers of a Matrix*] For nonnegative integers p and q , and square matrices A and B , we have

1. $A^p A^q = A^{p+q}$
2. $(A^p)^q = A^{pq}$

Note that in general $(AB)^p \neq A^p B^p$. This is precisely because commutativity does not hold for matrix multiplication.

Theorem 5 [*Properties of Scalar Multiplication*] Let r and s be real numbers and A and B be matrices of appropriate sizes. Then

1. $r(sA) = (rs)A$
2. $(r + s)A = rA + sA$
3. $r(A + B) = rA + rB$
4. $A(rB) = r(AB) = (rA)B$

Theorem 6 [Properties of Transpose] Let A and B be matrices of appropriate sizes and let r be a scalar. Then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(rA)^T = rA^T$

Finally, we define some special square matrices.

Definition 2 A square matrix A is said to be **involutory** if $A^2 = I$.

Definition 3 A square matrix A is said to be **periodic of period k** if $A^{k+1} = A$. In particular, A is said to be **idempotent** if $A^2 = A$.

Definition 4 A square matrix A is said to be **nilpotent** if $A^k = 0$ for some positive number k . The smallest such k is called the **index**.

Definition 5 A square matrix A is said to be **symmetric** if $A^T = A$. The square matrix is said to be **skew-symmetric** if $A^T = -A$.

Exercises:

1. Prove Theorem 1.
2. Prove Theorem 2.
3. Prove Theorem 4.
4. Prove Theorem 5.
5. Prove Theorem 6.
6. Show that $(-1)A = -A$.
7. Give an example of two non-zero matrices A and B for which $AB = 0$. (Note that for real numbers a and b , $ab = 0$ would automatically mean $a = 0$ or $b = 0$. This not true for matrices.)

8. Show that $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

9. Show $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent of index 3.

10. Find all 2×2 idempotent matrices.
11. Find all 2×2 symmetric matrices.
12. Find all 2×2 skew-symmetric matrices.