Math 114 Lecture: Matrices and its Operations

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Lecture 2

One purpose of this course is to explore another form of solving systems of linear equations. We now introduce the concept of a matrix.

Definition 1 *An $m \times n$ matrix A is a rectangular array of mn real (or complex) numbers arranged in m horizontal rows and n vertical columns:*

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The i-th row of matrix A is

$$[a_{i1}\ a_{i2}\ \cdots\ a_{in}] \text{ where } 1 \leq i \leq m.$$

The j-th row of A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \text{ where } 1 \leq j \leq n.$$

If $m = n$, then we say that A is a *square matrix.* A square matrix $A = [a_{ij}]$ for which every term off the main diagonal is zero, i.e. $a_{ij} = 0, i \neq j$, is called a *diagonal matrix.* A diagonal matrix for which all diagonal terms are equal, i.e. $a_{ij} = c$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$, is called a *scalar matrix.* A scalar matrix for which $c = 1$ is called the *identity matrix* often denoted by I. In particular, an identity matrix of size $n \times n$ is denoted by I_n.

Definition 2 *Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be* **equal** *if $a_{ij} = b_{ij}$, for every $1 \leq i \leq m$ and $1 \leq j \leq n$.*

Example: Consider the following matrices A and B,

$$A = \begin{bmatrix} a & 1 & -2 \\ 3 & 0 & 2 \\ 1 & 1 & c \end{bmatrix} \quad , \quad B = \begin{bmatrix} 6 & 1 & -2 \\ a+b & 0 & b+d \\ 1 & 1 & 2 \end{bmatrix}.$$

The two matrices are equal only if $a = 6$, $b = -3$, $c = 2$ and $d = 5$.

Operations on Matrices

Definition 3 *Let $A = [a_{ij}]$ and $B = [a_{ij}]$ be a two $m \times n$ matrices. The* **sum** *of A and B is the $m \times n$ matrix $C = [c_{ij}]$ defined by*

$$c_{ij} = a_{ij} + b_{ij}, \qquad 1 \leq i \leq m, 1 \leq j \leq n.$$

Definition 4 *Let $A = [a_{ij}]$ be an $m \times p$ matrix and $B = [a_{ij}]$ be a $p \times n$ matrix. The* **product** *of A and B is the $m \times n$ matrix $C = [c_{ij}]$ defined by*

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}, \qquad 1 \le i \le m, 1 \le j \le n.$$

Remarks:

1. BA may not be defined since the number of columns of B may not be equal to the number of rows of A. Thus, the product BA may be defined only if $m = n$.
2. If BA is defined, which means $m = n$, then BA is of size $p \times p$. On the other hand, AB would be of size $n \times n$. Thus, in general, $AB \neq BA$ on the basis of their sizes.
3. Even if AB and BA are of the same size, they still may or may not be equal. This implies matrix multiplication is not commutative.

Definition 5 *Let A be a square matrix. Then if p is a positive integer, we define the* **p-th power** *of A by*

$$A^p = \underbrace{A \cdot A \cdots A}_{p \text{ factors}}.$$

Definition 6 *Let $A = [a_{ij}]$ be an $m \times n$ matrix and let r is any real number. The* **scalar multiple** *of A by r, denoted by rA, is the $m \times n$ matrix $[b_{ij}]$ where $b_{ij} = ra_{ij}$, $1 \le i \le m$ and $1 \le j \le n$.*

Definition 7 *Let $A = [a_{ij}]$ be an $m \times n$ matrix. The* **transpose** *of A, denoted by A^T, is the $n \times m$ matrix $[b_{ij}]$ such that $b_{ij} = a_{ji}$. That is, the transpose of A is obtained by interchanging the rows and columns of A.*

Exercises:

1. Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & -2 \end{bmatrix}$$

   If possible, compute for:

   (a) $A + B$
   (b) $B + (-1)C$
   (c) $(AD)E$
   (d) $A(DE)$
   (e) $D + F^T$

2. Show why, in general, $(A \pm B)^2 \neq A^2 \pm 2AB + B^2$ and $(A + B)(A - B) \neq A^2 - B^2$.

3. Given

$$A == \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

   (a) Show $AB = BA = 0$. What can we conclude from this computation?
   (b) Show that $AC = A$ and $CA = C$. What can we conclude from this computation?
   (c) For this particular case, can we say $(A \pm B)^2 \neq A^2 + B^2$ and $(A + B)(A - B) \neq A^2 - B^2$?