Math 114 Lecture: Orthogonal Diagonalization of Symmetric Matrices

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Lecture 19

Given an $n \times n$ diagonalizable matrix A, can we choose a matrix P such that the colomns of P form an orthonormal set in $\mathbb{R}^n$?

Definition 1 *A nonsingular $n \times n$ matrix A is said to be an* **orthogonal matrix** *if*

$$A^{-1} = A^T \qquad \text{or equivalently,} \qquad AA^T = I_n.$$

Theorem 1 *An $n \times n$ matrix A is an orthogonal matrix if and only if the columns of A form an orthonormal set of vectors in $mathbbR^n$.*

If we choose an orthogonal matrix P which diagonalizes A, we say that A is **orthogonally diagonalizable**. The following theorem characterizes which matrices are orthogonally diagonalizable.

Theorem 2 *If A is an $n \times n$ matrix, then A is orthogonally diagonalizable if and only if A is symmetric.*

Theorem 3 *If A is a symmetric matrix, then:*

1. *the eigenvalues of A are all real numbers.*
2. *the eigenvectors associated with distinct eigenvalues are orthogonal vectors.*

Orthogonally diagonalizing a Symmetric Matrix

1. Find the eigenvalues and eigenspace associated to each eigenvalue of the symmetric matrix A.
2. Find a basis for each eigenspace of A.
3. Apply the Gram-Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.
4. Form the matrix P whose columns are the orthonormal vectors found in the previous step. Then P will be an orthogonal matrix ($P^{-1} = P^T$) and $P^T AP = D$, a diagonal matrix whose entries are the corresponding eigenvalues of A.

Example: Orthogonally diagonalize $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

The characteristic equation $(\lambda - 2)^2 - 4 = 0$ gives eigenvalues $\lambda = 0, 4$. For $\lambda = 0$, the eigenvectors are $\left\{ \begin{bmatrix} a \\ -a \end{bmatrix}, a \in \mathbb{R} \right\}$ while for $\lambda = 4$, the eigenvectors are $\left\{ \begin{bmatrix} b \\ b \end{bmatrix} b \in \mathbb{R} \right\}$. A basis for $E(0)$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and a basis for $E(4)$ is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Applying the Gram-Schmidt process to each of the basis, we get $P = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$. Note that P is an orthogonal matrix (verify!) and so

$$P^{-1}AP = P^T AP = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}.$$

Exercises: Find a matrix P that orthogonally diagonalizes A.

1. $A = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

3. $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$

4. $A = \begin{bmatrix} -3 & 0 & -1 \\ 0 & -2 & 0 \\ -1 & 0 & -3 \end{bmatrix}$

Exercise: Use diagonalization to evaluate A^6 if $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.