Math 114 Lecture: Diagonalization of Matrices

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Lecture 18

We already know that given a linear transformation $L : V \to W$, we can find different matrix representations for L. We now look at how these matrices are related to each other.

Definition 1 *A matrix B is said to be* <u>similar</u> *to matrix A if thre is a non-singular matrix P such that* $\boxed{B = P^{-1}AP}$.

Example: If $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, find a matrix B similar to A, given the matrix P:

1. $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Then $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$. Thus

$$B = P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

2. $P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. Then $P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$. Thus

$$B = P^{-1}AP = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$$

Theorem 1 *Let V be any n-dimensional vectore space and let A and B be $n \times n$ matrices. Then A and B are similar if and only if they represent the same linear transformation $L : V \to V$ with respect to two ordered bases for V.*

Theorem 2 *Similarity is an equivalence relation. That is, for $n \times n$ matrices A, B and C,*

1. *A is similar to A.*
2. *If A is similar to B then B is similar to A.*
3. *If A is similar to B and B is similar to C, then A is similar to C.*

Definition 2 *The matrix A is* <u>diagonalizable</u> *if it is similar to a diagonal matrix.*

Theorem 3 *A square matrix A is said to be diagonalizable if and only if it has n linearly independent eigenvectors. In this case, $A = PDP^{-1}$, where D is a diagonal matrix consisting of the eigenvalues of A, while P is a matrix whose columns are the n linearly independent eigenvectors of A.*

When will a matrix have n linearly independent eigenvectors? Well, eigenvectors associated with distinct eigenvectors are linearly independent.

Theorem 4 *A matrix A is diagonalizable if all the roots of the characteristic polynomial are real and distinct.*

Now, what if A does not have n distinct real roots? Will A still be diagonalizable? The answer: it depends. If one can find n linearly independent eigenvectors, then A can still be diagonalizable. This will happen if the dimension of the eigenspace of each distinct real eigenvalue λ is equal to its mulitplicity as a root of the characteristic polynomial.

We now summarize the procedure of diagonalizaing a square matrix A.

Diagonalizing a matrix:

1. Form the characteristic polynomial $f(\lambda) = |\lambda I_n - A|$ of A.
2. Find the roots of $f(\lambda) = 0$. If there are complex roots, then A cannot be diagonalizable.
3. Let λ_i be an eigenvalue of A with multiplicity k_i.
4. Find a basis for the eigenspace associated with λ_i. If the $dim(\text{eigenspace},\lambda_i) < k_i$, then A is not diagonalizable.
5. Suppose $dim(E(\lambda_i)) = k_i$. Let P be the matrix whose columns are the linearly independed eigenvectors of A.
6. $P^{-1}AP = D$ where D is precisely the diagonal matrix whose diagonal entries are the eigenvalues of A.

In our previous example, with matrix $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$, we found the eigenspaces associated with the eigenvalues $\lambda = 0, 2$.

$$\left\{ \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}, x_2 \in \mathbb{R} \right\} \text{ for } \lambda = 0 \text{ and } \left\{ \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}, x_2 \in \mathbb{R} \right\} \text{ for } \lambda = 2$$

We form the matrix $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. The diagonal matrix similar to A is $D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

Why is it important to know if a matrix is diagonalizable? If A is diagonalizable, then A is similar to D where $D = P^{-1}AP$. Equivalently, $A = PDP^{-1}$ (why?). Now, for any positive integer k,

$$\begin{aligned} A^k = (PDP^{-1})^k &= \underbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{k \text{ factors}} \\ &= PD(PP^{-1})D(PP^{-1})\cdots(PP^{-1})DP^{-1} \\ &= PD^kP^{-1} \end{aligned}$$

That is, getting powers of a diagonalizable matrix A only needs to get the powers of the diagonal matrix (which is easy) and multiply P on the left and P^{-1} on the right. The computation in this case is relatively easy than actually getting A^k.

Exercises:

1. Determine if the given matrices are diagonalizable:
   (a) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
2. Find the matrix P such that $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = PDP^{-1}$.
3. Find the matrix P that diagonalizes $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$.
4. Determine if the following matrix is diagonalizable: $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$.