Math 114 Lecture: Eigenvalues and Eigenspaces

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Lecture 17

(For the rest of the discussions, all matrices are assumed to be square matrices.)

Definition 1 *Let A be an $n \times n$ matrix. The real number λ is called an eigenvalue of A if there exists a non-zero vector $X \in \mathbb{R}^n$ such that $AX = \lambda X$. All such vectors X are called eigenvectors of A associated with the eigenvalue λ. The set of all such eigenvectors is called the eigenspace $E(\lambda)$ associated with each eigenvalue λ.*

Note that if we want all vectors X satisfying $AX = \lambda X$ for a given λ then $AX - \lambda(X) = 0$ or $(A - \lambda I_n)X = 0$ where I_n is the identity matrix. Since the vectors X must be non-trivial (or non-zero), then that means $A - \lambda I_n$ should be singular, implying $det(A - \lambda I_n) = 0$.

Definition 2 *Let A be an $n \times n$ matrix. The determinant $f(\lambda) = |A - \lambda I_n| = |\lambda I_n - A|$ is called the characteristic polynomial of A. The equation $f(\lambda) = |\lambda I_n - A| = 0$ is called the characteristic equation of A.*

Theorem 1 *The eigenvalues of A are the real roots of the characteristic equation of A.*

Definition 3 *The dimension of an eigenspace $E(\lambda)$ is the number of all basis vectors of the eigenspace.*

Example: For matrix A, find the eigenvalues of A and find the basis of the eigenvectors associated with each of the eigenvalues determined.

Given $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ we have $\lambda I_n - A = \begin{bmatrix} \lambda - 1 & -1 \\ -1 & \lambda - 1 \end{bmatrix}$ so that

$$f(\lambda) = (\lambda - 1)^2 - 1 = \lambda^2 - 2\lambda = 0 \Longrightarrow \lambda = 0, 2$$

To get the eigenvectors associated with the eigenvalue $\lambda = 0$:

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow x_1 = -x_2$$

Thus the eigenspace associated with $\lambda = 0$ is

$$E(\lambda = 0) = \left\{ \begin{bmatrix} -x_2 \\ x_2 \end{bmatrix}, x_2 \in \mathbb{R} \right\} \text{ with a basis represented by} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

This gives the dimension $dim\ E(\lambda = 0) = 1$.

To get the eigenvectors associated with the eigenvalue $\lambda = 2$:

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow x_1 = x_2$$

Thus the eigenspace associated with $\lambda = 2$ is

$$E(\lambda = 2) = \left\{ \begin{bmatrix} x_2 \\ x_2 \end{bmatrix}, x_2 \in \mathbb{R} \right\} \text{ with a basis represented by} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

This gives the dimension $dim\ E(\lambda = 2) = 1$.

To summarize, here are the steps in finding the eigenvalues and eigenvectors associated with each eigenvalues, given an $n \times n$ matrix A:

1. Form the matrix $\lambda I_n - A$.
2. Find the determinant of $\lambda I_n - A$. This gives the characteristic polynomial of A.
3. Determine the zeroes of the characteristic equation of A, $|\lambda I_n - A| = 0$. The zeroes are precisely the eigenvalues of the matrix.
4. For each eigenvalue λ, find the eigenspace associated with λ by solving for the vectors satisfying the equation $AX = \lambda X$.

Exercise: Given the matrix of linear transformation A, find the eigenvalues of A and find a basis for the eigenspace associated with each of the eigenvalues determined.

1. $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
2. $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$
3. $A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ 0 & 4 & 3 \end{bmatrix}$

Exercise: Find the eigenvalues and eigenspaces of the following linear operator L on P_1 defined by $L(at + b) = (a + 2b)t + (4a + 3b)$.