

Math 114 Lecture: Matrix of a Linear Transformations

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Lecture 16

Given a basis $S = \{v_1, v_2, \dots, v_n\}$ for V , we know that as a set, the order in which the vectors appear does not change the set. We now talk of an ordered basis for V , where the order in which the vectors appear matters. For example, the ordered basis $\{v_1, v_2, v_3\}$ is not the same as the ordered basis $\{v_2, v_1, v_3\}$.

Definition 1 Let V be a vector space and let $S = \{v_1, v_2, \dots, v_n\}$ be an ordered basis for a real vector space V . If $v \in V$, then v can be written uniquely as

$$v = a_1v_1 + a_2v_2 + \dots + a_nv_n.$$

The coordinate vector of v with respect to S is

$$[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}.$$

Exercises:

1. Let $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Find the coordinate vector of $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$.
2. Let $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$. Find the coordinate vector of $\begin{bmatrix} 4 \\ -2 \end{bmatrix}$.
3. For P_2 , let $S = \{t^2 - t + 1, t + 1, t^2 + 1\}$. Find the coordinate vector of $v = 4t^2 - 2t + 3$.
4. For P_2 , let $S = \{t^2 + 1, t + 1, t^2 + t\}$ and let $[v]_S = [3 \ -1 \ -2]^T$. Find v .

Matrix of Linear Transformation

Definition 2 Let $L : V \rightarrow W$ be a linear transformation and let $S = \{v_1, v_2, \dots, v_n\}$ and $T = \{w_1, w_2, \dots, w_m\}$ be ordered bases for V and W , respectively. For each v_i , compute the coordinate vector of $L(v_i)$ with respect to T . Then the matrix of L with respect to S and T is

$$A = [[L(v_1)]_T \quad [L(v_2)]_T \quad \dots \quad [L(v_n)]_T].$$

Moreover, for all $v \in V$, $[L(v)]_T = A[v]_S$.

Examples:

1. Let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be defined by $L \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y \\ y - z \end{bmatrix}$, with ordered bases S and T as follows:
 $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Find the matrix of L with respect to S and T .

2. Given the linear transformation in number 1, find the matrix of L with respect to S , the standard basis for \mathbb{R}^3 and $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Use matrix A to find $L \left(\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \right)$.
3. Let $L : P_1 \rightarrow P_2$ be defined by $L(p(t)) = tp(t)$. Let $S = \{1, t\}$ and $T = \{t^2, t-1, t+1\}$ be ordered bases for P_1 and P_2 , respectively. Find the matrix of L with respect to S and T .
4. Let $L : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a linear transformation and let $S = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be ordered bases and $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$ is the matrix of L with respect to S and T .
Find (a) $L \left(\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right)$ (b) $L \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right)$

Isomorphisms(Optional Topic)

Definition 3 A linear transformation from $L : V \rightarrow W$ is called an isomorphism if L is one-to-one and onto. If such a linear transformation exists between two vector spaces V and W , then we say V is isomorphic to W .

Examples:

- The linear transformation from $L : M_{22} \rightarrow \mathbb{R}^4$ given by $L \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = [a \ b \ c \ d]^T$ is an isomorphism.
- The linear transformation from $L : P_2 \rightarrow \mathbb{R}^3$ given by $L(at^2+bt+c) = [a \ b \ c]^T$ is an isomorphism.

Theorem 1 Every n -dimensional vector space V is isomorphic to \mathbb{R}^n .

Proof: Let $S = \{v_1, \dots, v_n\}$ be any ordered basis for V . Define the linear transformation $L : V \rightarrow \mathbb{R}^n$ by $L(v) = [v]_S$, the coordinate vector of v with respect to S . Then L is an isomorphism.

Remark: If V and W are isomorphic, then the algebraic properties of V and W are the same, only the nature of the vectors differ. For example, in P_2 , adding the polynomials $3t^2 - 1$ and $t^2 + 2t + 3$ is similar to adding their coordinate vectors with respect to the ordered basis $\{t^2, t, 1\}$, $[3 \ 0 \ -1]^T$ and $[1 \ 2 \ 3]^T$ in \mathbb{R}^3 .