Math 114 Lecture: Linear Transformations

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Lecture 15

Definition 1 *Let V and W be vector spaces over the same field $\mathcal{F}$. A linear transformation L of V into W is a function assigning a unique vector $L(x)$ in W for each x in V such that:*

1. $L(x+y) = L(x) + L(y) \quad \forall x, y \in V$.
2. $L(cx) = cL(x) \quad \forall x \in V$ *and c is a scalar.*

Remark: If $V = W$, then the transformation map $L : V \to W$ is called a linear operator.

Examples:

1. $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $(x, y, z) \longmapsto (x + y, z)$
2. $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $(x, y, z) \longmapsto (x, y)$ (projection)
3. $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $(x, y, z) \longmapsto (cx, cy, cz)$ (contraction/dilation)
4. $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(x, y) \longmapsto (x, -y)$ (reflection wrt x-axis)
5. $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(x, y) \longmapsto (-x, y)$ (reflection wrt y-axis)
6. $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(x, y) \longmapsto (ca + x, ab + y)$ (translation)
7. $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ (rotation by θ)
8. $L : C[0, 1] \to \mathbb{R}$ such that $L(f) = \int_0^1 f(x)dx$ (integral transform)

Remark: If f is a linear transformation then $L(0_V) = 0_W$ where 0_V and 0_W are the zero vectors of V and W, respectively.

Theorem 1 *Let $L : V \to W$ be a linear transformation of an n-dimensional vector space V into a vector space W. Let $S = \{x_1, x_2, \ldots, x_n\}$ be a basis for V. If x is any vector in V, then $L(x)$ is completely determined by $\{L(x_1), L(x_2), \ldots, L(x_n)\}$.*

Remark: Even if $\{x_1, x_2, \ldots, x_n\}$ is a basis for V, it does not follow that $\{L(x_1), L(x_2), \ldots, L(x_n)\}$ is also a basis for W.
Exercise: Show that the trace of an $n \times n$ matrix A, $tr(A)$, is a linear transformation. Note that the trace tr is a mapping from M_n to $\mathbb{R}$.

Kernel and Range of a Linear Transformation

Definition 2 *A linear transformation $L : V \to W$ is one-to-one if for all $X_1, X_2 \in V$, $X_1 \neq X_2$ implies $L(X_1) \neq L(X_2)$. Equivalently, L is 1-1 if for $X_1, X_2 \in V$, $L(X_1) = L(X_2)$ implies $X_1 = X_2$.*

Exercise: Determine whether the linear transformation is one-to-one or not.

1. $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L(x, y) = (x + y, x - y)$
2. $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(x, y, z) = (x, z)$

Definition 3 *Let $L : V \to W$ be a linear transformation. The kernel of L, $kerL$, is the subset of V consisting of X such that $L(X) = 0_W$. That is, all vectors in this subset of V is mapped to the 0 in W.*

Exercise: Find the kernel of each linear transformation.

1. $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(x, y, z) = (x, z)$
2. $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L(x, y) = (x + y, x - y)$
3. $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that $L \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

Theorem 2 *Let $L : V \to W$ be a linear transformation. Then $kerL$ is a subspace of V.*

Theorem 3 *If $kerL = \{0\}$ then the dim $kerL = 0$.*

Theorem 4 *$L : V \to W$ is one-to-one if and only if $kerL = 0_V$.*

Definition 4 *If $L : V \to W$ is a linear transformation, then the range of L, rangeL, is the set of all vectors in W that are images of vectors in V under L. We say vector Y is in rangeL if we can find some vector X in V such that $L(X) = Y$. If $rangeL = W$, then L is said to be onto.*

Theorem 5 *Let $L : V \to W$ be a linear transformation. Then $rangeL$ is a subspace of W.*

Exercise: Determine whether L is onto. Also find the $dim(rangeL)$.

1. . $L : \mathbb{R}^3 \to \mathbb{R}^2$ such that $L(x, y, z) = (x, y)$
2. $L : \mathbb{R}^3 \to \mathbb{R}^3$ such that $L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Theorem 6 *If $L : V \to W$ is a linear transformation, then*

$$dim(kerL) + dim(rangeL) = dim\ V.$$

Remark: The $dim(kerL)$ is also called the *nullity* of L.

Exercises:

1. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.
   (a) Is L onto?
   (b) Find a basis for the range of L.
   (c) Find teh kernel of L.
2. Let $\mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by
   $$L([a\ b\ c\ d]) = [a + b \quad c + d \quad a + c].$$
   (a) Find a besis for *ker* L. What is *dim ker* L?
   (b) Find a basis for *range* L? What is *dim range* L?
3. Let $L : P_2 \to P_3$ be the linear transformation defined by $L[p(t)] = t^2 p'(t)$, where $p'(t)$ is the derivative of $p(t)$ with respect to t.
   (a) Find a basis for *ker* L.
   (b) Is L onto? Explain your answer.

Corollary 1 *Let $L : V \to W$ be a linear transformation, and let $dim\ V = dim\ W$. Then,*

1. *if L is 1-1, then it is onto.*
2. *if L is onto, then it is 1-1.*