Math 114 Lecture: Orthonormal Bases

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Lecture 14

Definition 1 *A set $S = \{v_1, v_2, \ldots, v_n\}$ is called orthogonal if any two distinct vectors in S are orthogonal, that is $< u_i, u_j >= 0$ whenever $i \neq j$. Furthermore, if each vector is a unit vector, that is, it has length 1, then the set is said to be orthonormal.*

Examples:

- The set S $= \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ is an orthogonal set of vectors, but it is not an orthonormal set.
- The set T $= \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}, \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix} \right\}$ is an orthonormal set in $\mathbb{R}^3$ and is actually a basis for $\mathbb{R}^3$.

Theorem 1 *Let $S = \{v_1, v_2, \ldots, v_n\}$ be set of orthonormal vectors in an inner product space V. Then every vector $u \in V$ can be written as*

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n,$$

where $\quad a_i =< u, v_i >$.

Example: We already know that $v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -\frac{4}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$, $v_3 = \begin{bmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{bmatrix}$ form an orthonormal basis for $\mathbb{R}^3$. Consider $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Note that

$$\begin{aligned} < u, v_1 > &= 2 \\ < u, v_2 > &= 1 \\ < u, v_2 > &= 3 \end{aligned}$$

Therefore, $u = 2v_1 + v_2 + 3v_3$.

<u>Gram-Schmidt Process</u>

Let V be an inner product space V and $W \neq \overrightarrow{v}$ an m-dimensional subspace of V. then there exists an orthonormal basis $T = \{w_1, w_2, \ldots, w_n\}$ for W.

<u>How to find an orthonormal basis for W :</u> Let $\{u_1, u_2, \ldots, u_n$ be any basis for W.

1. First, we'll look for an orthogonal basis for W. Let

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= u_2 - \frac{< u_2, v_1 >}{< v_1, v_1 >} v_1 \\ &\vdots \\ v_m &= u_m - \sum \frac{< u_m, v_i >}{< v_i, v_i >} v_i \end{aligned}$$

   Then the set $\{v_1, v_2, \ldots, v_m\}$ is an orthogonal basis for W.

2. Let $z_i = \dfrac{v_i}{\|v_i\|}$. Then every z_i if of length 1. Therefore the set $\{z_1, z_2, \ldots, z_m\}$ is an orthonormal basis for W.

Examples:

1. Transform the basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ into an orthonormal basis for $\mathbb{R}^3$.

2. Find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$.

3. Find an orthonormal basis for $W = \left\{ \begin{bmatrix} a+c \\ a+b+2c \;\; c \\ b+c \end{bmatrix} \right\}$.

<u>Orthogonal Complements</u>

Given any subset W of $\mathbb{R}^n$, we can find another subset V of $\mathbb{R}^n$ such that every element of $\mathbb{R}^n$ can be written uniquely as $w + v$, where $w \in W$ and $v \in V$. For example, consider

$$W = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}.$$

We can find that if $V = \left\{ \begin{bmatrix} 0 \\ b \\ c \end{bmatrix} : a \in \mathbb{R} \right\}$, then we have the desired result. This set V is called the orthogonal complement of W in $\mathbb{R}^n$.

Definition 2 *Let W be a subspace of $\mathbb{R}^n$. A vector $u \in \mathbb{R}^n$ is said to be orthogonal to W if it is orthogonal to every vector in W. The set of all vectors in $\mathbb{R}^n$ that are orthogonal to all the vectors in W is called the* **orthogonal complement** *of W in $\mathbb{R}^n$ and is denoted by $W^\perp$.*

Theorem 2 *L*

et W be a subspace of $\mathbb{R}^n$. Then

1. $W^\perp$ is a subspace of $\mathbb{R}^n$.

2. $W \cap W^\perp = \{O_v\}$.

3. Every vector v in $\mathbb{R}^n$ can be written uniquely as $w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^\perp$.

Examples

1. Find the orthogonal complement of $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$. Find a basis for it.

2. Find a basis for $W^\perp$, if W has basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$.