Math 114 Lecture: Basis and Dimension

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Lecture 13

Definition 1 *A set S in V is a* **basis** *for V if and only if S is linearly independent that spans V.*

Examples:

1. The set $\{(1,0),(1,1)\}$ is a basis for $\mathbb{R}^3$.
2. In P_2, $\{t^2, t, 1\}$ is a basis. Similarly, the set $\{t^n, t^{n-1}, , t, 1\}$ is a basis for P_n.
3. In M_{22}, the set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis.

Exercises:

1. If $S = \{X_1, X_2, X_3, X_4\}$ where $X_1 = (1,1,0)$, $X_2 = (1,0,1)$ and $X_3 = (1,1,1)$, does S span $\mathbb{R}^3$? Is S linearly independent? Does S form a basis for $\mathbb{R}^3$?
2. Does $(3,2,2)$, $(-1,2,1)$ and $(0,1,0)$ form a basis for $\mathbb{R}^3$?
3. Does $3t^2 + 2t + 1$, $t^2 + t + 1$ and $t^2 + 1$ form a basis for P_2, the vector space of all second-degree polynomials in t?

Remarks:

1. The basis of a vector space is not unique.
2. If e_i is an n-vector defined by $e_i = (0, .., \underbrace{1}_{i^{th} \text{ entry}}, .., 0)$ then the set $S = \{e_1, e_2, \dots, e_n\}$ forms a basis for $\mathbb{R}^n$. In this case S is called the **natural basis** for $\mathbb{R}^n$.

Theorem 1 *If S is a basis for V, then every vector in V can uniquely be represented as a linear combination of S.*

We have mentioned that a basis for a vector space is not unique. However, every basis for a vector space has the same number of elements.

Definition 2 *The* **dimension** *of a non-zero vector space V, denoted by dim V, is the number of vectors in a basis for V.*

The set containing only the zero vector is defined to have dimension 0.
Examples:

1. $dim\ \mathbb{R}^n = n$
2. $dim\ P_n = n + 1$
3. $dim\ M_{22} = 4$
4. $dim\ M_{mn} = mn$

Remarks: Suppose V is vector space of dimension n.

1. A largest linearly independent subset of V contains n vectors and is a basis for V.
2. Any subset of V containing more than n vectors is linearly dependent.
3. A smallest spanning set for V contains n vectors and is a basis for V.
4. Any subset of V containing less than n vectors cannot span V.

If V is a finite dimensional vector space and we have a set of linearly independent vectors in V, then we asked earlier whether we can obtain a basis from this linearly independent set. We can now state the following theorem.

Theorem 2 *If S is a linearly independent set of vectors in a finite-dimensional vector space V, then there is a basis T for V that contains S.*

Proof:
Let S = be a linearly independent set in a finite-dimensional vector space V, where k ¡ n. Let be a known basis for V. Then the set spans V, hence contains a basis for V. We check whether wm is a linear combination of the preceding vectors and if it is, we delete it. We do this process repeatedly for wm-1,, w2, and w1 to obtain the desired basis T. Note that none of the vI's will be deleted since they form a linearly independent set. Hence, the basis that we obtain in this manner will contain S.

Theorem 3 *Let V be an n-dimensional vector space, and let $S = \{X_1, X_2, \ldots, X_n\}$ be a set of n vectors in V.*

1. *If S is linearly independent, then S is a basis for V.*
2. *If S spans V, then S is a basis for V.*

Question: Suppose that the number of entries in a vector (m)is less than the number of vectors (n) given , how do we find a basis for the m-dimensional space V?

Solution: Let $S = \{X_1, X_2, \ldots, X_k\}$ be a spanning set for $V \leq \mathbb{R}^n$, $k \leq n$. Then a basis for V is derived as follows:

1. Form the matrix $A = [X_1 X_2 \cdots X_k]^T$.
2. Transform A to its (reduced) row echelon form (rref).
3. Determine the rank of A by counting the number of nonzero rows of the (r)ref of A.
4. The rank of A determines the number of basis vectors of the spanning set of S.

Example: Consider the following elements of S:

$$X_1 = (1, 1, 0, -1) \quad X_2 = (1, 2, 3, 4) \quad X_3 = (2, 3, 3, 3).$$

Find a basis for the spanning set of S.

Solution: Form the matrix A first, and transform it into its reduced row echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & 3 & 5 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of A is 2. Thus the number of basis vectors in the spanning set of S is 2. The basis is therefore any two of the vectors in S.