Math 114 Lecture: Span and Linear Independence

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Lecture 12

Linear Combination

Note that almost all of the vector spaces we have dealt with previously are infinite, that is, they have an infinite number of elements. We now look at vector spaces which can be completely described using only a finite set of vectors.

Definition 1 *Let $v_1, v_2, \ldots, v_k$ be a set of vectors in a vector space V. A vector $v \in V$ is said to be a linear combination of the vectors if there exist scalars $c_1, c_2, \ldots, c_k$ such that*

$$v = c_1v_1 + c_2v_2 + \ldots + c_kv_k = \sum_{i=1}^{k} c_iv_i.$$

Examples:

1. In $\mathbb{R}^3$, let $v_1 = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} v_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$. Then $v = \begin{bmatrix} -6 \\ -3 \\ -1 \end{bmatrix}$ is a linear combination of v_1, v_2, v_3 since $v = (-1)v_1 + 2v_2 + 3v_3$.
2. Every vector v is a linear combination of itself since $v = 1v$.
3. The zero vector is a linear combination of any set of vectors $v_1, v_2, \ldots, v_k$, since $0 = 0v_1 + 0v_2 + \ldots + 0v_n$.

How does one determine if a vector v is a linear combination of the vectors $v_1, v_2, \ldots, v_k$?

1. Form the linear system $v = c_1v_1 + c_2v_2 + \ldots + c_kv_k$ with unknowns c_i's.
2. Solve the linear system. If the system has a solution, then v is a linear combination of the vectors.

Remark: Note that if v is a linear combination of a set of vectors, then there exist scalars c_i's such that $v = c_1v_1 + c_2v_2 + \ldots + c_kv_k$ but the c_i's need not be unique. Hence, there may be several ways of expressing a vector v as a linear combination of a set of vectors.

Exercises: Determine if the given vector is a linear combination of the given vectors v_i's.

1. In $\mathbb{R}_3$, is $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ a linear combinations of $v_1 = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$?
2. In P_2, is $p(t) = 3t^2 + 2t - 1$ a linear combination of $v_1 = t^2 - t$, $v_2 = t^2 - 2t + 1$, $v_3 = -2t^2 + 1$?

Span of a Vector/Subspace

Definition 2 *Let S be a nonempty subset of a vector space V. The set W consisting of all linear combinations of elements of S is called the span of S.*

Example: Is $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ in the span of $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$?
Solution: We want to know if there exist c_1, c_2 such that

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 3 \end{bmatrix} \Longrightarrow \left[\begin{array}{cc:c} 1 & -1 & 1 \\ 2 & 3 & 3 \end{array}\right].$$

Row reducing, this system has the solution $c_1 = 6/5$ and $c_2 = 1/5$, hence, $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$.

Theorem 1 *The span S is a subspace of V.*

Definition 3 *The vectors $v_1, v_2, \ldots, v_k$ in a vector space V are said to span V if every vector in V can be expressed as a linear combination of the vectors. Moreover, if $S = \{v_1, v_2, \ldots, v_k\}$, then we also say that S spans V or V is spanned by S.*

Exercises:

1. Does $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\}$ span $\mathbb{R}^2$?
2. Does $W = \{t^2 + 2t - 1, t^2 - 1\}$ span P_2?

Linear Independence

Given any set of vectors $v_1, v_2, \ldots, v_k$, we already know that the zero vector is always a linear combination of these vectors since we can take $a_i = 0$ for all $i = 1, 2, \ldots, n$. But is it still possible to express the zero vector as a linear combination of the v_i's where not all the scalars are zero? We look at two examples in $\mathbb{R}^2$.

Example 1: Let $v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$. If we let $0 = a_1 v_1 + a_2 v_2$, solving the system will give us $a_1 = a_2 = 0$.
Example 2: Let $v_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$. If we let $0 = a_1 v_1 + a_2 v_2$, then the system will have infinitely many solutions, in particular, we can take $a_1 = 2$ and $a_2 = 1$.

We can now define the concept of linear dependence and independence.

Definition 4 *Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vectors in a vector space V. Then S is said to be linearly dependent if there exist scalars $a_1, a_2, \ldots, a_k$, not all zero, such that $0 = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k$. Otherwise, S is said to be linearly independent. That is, a set is linearly independent if whenever $0 = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k$, then $a_i = 0$ for all i.*

How does one determine if a set of vectors is linearly dependent/independent?

1. Form the linear system $0 = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k$ with unknowns a_i's.
2. Solve the homogeneous system. If the solution is the trivial solution, then the vectors are linearly independent, otherwise, the set is linearly dependent.

In the previous examples, the vectors in example 1 are linearly independent while the vectors in example 2 are linearly dependent.

Remarks regarding linear independence:

1. Every set containing the zero vector is linearly dependent.
2. Every set $\{v\}, v \neq 0$, is linearly independent.
3. If S_1 and S_2 are finite subsets of a vector space such that $S_1 \subset S_2$,
   (a) If S_1 is linearly dependent, then so is S_2.
   (b) If S_2 is linearly independent, then so is S_1.
4. A set $S = \{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if at least one of the vectors in S is a linear combination of the preceding vectors.