Math 114 Lecture: Vector Spaces

Lecturer: Jose Maria L. Escaner IV, Ph.D.
Lecture 11

Definition 1 *A real vector space is a set V of elements (called vectors) on which we have two operations $\oplus$ and $\otimes$ defined with the following properties:*

1. *If u and v are any vectors in V, then $u \oplus v$ is in V.*
   (a) *$u \oplus v = v \oplus u$ for all $u, v \in V$.*
   (b) *$u \oplus (v \oplus w = (u \oplus v) \oplus w$ for all $u, v, w \in V$.*
   (c) *There exists an element $0 \in V$ such that $u \oplus 0 = 0 \oplus u = u$ for any $u \in V$. 0 is called the zero vector of V.*
   (d) *For each $u \in V$ there exists an element $-u \in V$ such that $u \oplus -u = -u \oplus u = 0$. $-u$ is called the negative of vector u.*
2. *If u is any vector in V and c is any real number (scalar), then $c \otimes u$ is in V.*
   (a) *$c \otimes (u \oplus v) = c \otimes u \oplus c \otimes v$ for any $u, v \in V$ and c any scalar.*
   (b) *$(c + d) \otimes u = c \otimes u \oplus d \otimes u$ for any $u \in V$ and c, d any scalars.*
   (c) *$c \otimes (d \otimes u) = (cd) \otimes u$ for any $u \in V$ and any scalars c and d.*
   (d) *$1 \otimes u = u$ for any u in V.*

Examples:

1. The set of all $n \times 1$ vectors, $\mathbb{R}^n$, with the operations vector addition and scalar multiplication is a real vector space.
2. The set of all $m \times n$ matrices, M_{mn}, with the operations matrix addition and matrix scalar multiplication, is a real vector space.
3. Let P_2 be the set of all polynomials in a variable, say t, with degree less than or equal to 2, that is $P_2 = \{at^2 + bt + c | a, b, c \in \mathbb{R}\}$ with the following operations:
   $$(at^2 + bt + c) \oplus (dt^2 + et + f) = (a + c)t^2 + (b + e)t + (c + f)$$
   $$r \otimes (at^2 + bt + c) = rat^2 + rbt + rc.$$
   Then P_2 is a real vector space under the given operations. Note that $\oplus$ and $\otimes$ are simply addition and multiplication on polynomials.
4. Let V be the set of all real-valued continuous functions defined on $\mathbb{R}$, that is, with the operations $(f \oplus g)(t) = f(t) + g(t)$ and $(c \otimes f)(t) = cf(t)$. Then V is a vector space and is denoted by $C(-\infty, +\infty)$.

<u>Some Properties of a Vector Space:</u>

1. $0 \otimes u = 0$ for any vector u in V.
2. $c \otimes 0 = 0$ for any scalar c.
3. If $c \otimes u = 0$, then either $c = 0$ or $u = 0$.
4. $(-1) \otimes u = -u$.

Exercise: Explain why the following are not vector spaces under the given operations.

1. The set of all ordered pairs of real numbers with the operations $(x, y) \oplus (x', y') = (x + x', y + y')$ and $c \otimes (x, y) = (x, cy)$.
2. The set of all ordered triples of real numbers with the operations $(x, y, z) \oplus (x', y', z') = (x + x', y + y', z + z')$ and $c \otimes (x, y) = (x, 1, z)$.
3. The set of all polynomials $p(t)$ with degree $= 2$ under the operations of P_2.
4. The set of all positive real numbers with operations $u \oplus v = uv - 1$ and $c \otimes u = u$.

Exercise: Let $V = \{x \in \mathbb{R} | x > 0\}$ with operations $x \oplus y = xy$ and $c \otimes x = xc$. Is V a real vector space? Prove or disprove.

SUBSPACES

Definition 2 *Let V be a given vector space and let W be a nonempty subset of V. If W is itself a vector space with respect to the operations in V, then W is said to be a subspace of V.*

Examples:

1. Every vector space V has at least two subspaces: itself and the set containing the zero vector, called the trivial subspace.
2. P_1, the set of all polynomials with degree less than or equal to 1, is a subspace of P_2, the set of all polynomials with degree less than or equal to 2.
3. The set of all symmetric 2x2 matrices is a subspace of M_{22}, the set of all 2×2 matrices.

If we have a subset of a given vector space and we want to know if it is a subspace, do we need to check whether it satisfies all the ten conditions for a vector space? The following theorem answers this.

Theorem 1 *Let W be a subset of a vector space V. Then W is a subspace of V if and only if for any $u, v \in W$ and c a scalar,*

$$cu + v \in W.$$

Exercise: Determine if W is a subspace of V.

1. Let $V = \mathbb{R}^3$. Let W be the set of all vectors of the form:
   (a) $[a\ b\ 1]^T$ (b) $[a\ b\ (a + 2b)]^T$ (c) $[a\ b\ c]^T, a > 0$
2. Let $V = M_{23}$. Let W be the set of all matrices of the form:
   (a) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = -2c$ and $f = 2e + d$
   (b) $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$, where $a = 2c + 1$
3. Let V be the set of all $n \times n$ matrices. Let W be:
   (a) the set of all diagonal matrices
   (b) the set of all nonsingular matrices
   (c) the set of all upper triangular matrices