Cofactor Expansion

Definition: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Let M_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting the i^{th} row and j^{th} column of A. The determinant of M_{ij} is called the minor of a_{ij}. The cofactor A_{ij} of a_{ij} is

$$\boxed{A_{ij} = (-1)^{i+j}|M_{ij}|.}$$

Theorem: Let $A = [a_{ij}]$ be an $n \times n$ matrix. Then for each $1 \le i \le n$, the cofactor expansion about the i^{th} row

$$\boxed{|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{k=1}^{n} a_{ik}A_{ik}.}$$

Also for $1 \le j \le n$, the cofactor expansion about the j^{th} column

$$\boxed{|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{k=1}^{n} a_{kj}A_{kj}.}$$

Examples:

1. $\begin{vmatrix} 2 & 1 & -3 \\ 0 & 1 & 2 \\ -4 & 2 & 1 \end{vmatrix}$ 2. $\begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}$ 3. $\begin{vmatrix} 2 & 2 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 3 & -1 & 4 & 1 \\ 2 & 3 & 0 & 0 \end{vmatrix}$

APPLICATIONS:

Finding the inverse of a Matrix using determinants

Definition: Let $A = [a_{ij}]$ be an $n \times n$ matrix. The adjoint of A, denoted by $adj A$, is the $n \times n$ matrix whose ij^{th} element is the cofactor A_{ji} of a_{ij}.

Example: Compute for the adjoint of:

$$\text{(a)}\ \begin{vmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{vmatrix} \qquad \text{(b)}\ \begin{vmatrix} 6 & 2 & 8 \\ -3 & 4 & 1 \\ 4 & -4 & 5 \end{vmatrix}$$

Theorem: If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$A(adj\,A) = (adj\,A)A = |A|I_n.$$

Corollary: If A is an $n \times n$ matrix and $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|}(adj\,A).$$

Theorem: A matrix A is nonsingular if and only if $|A| \neq 0$.

Corollary: $AX = 0$ has a non-trivial solution if and only if $|A| = 0$.

Examples: Determine whether the following matrix is non-singular:

$$1.\ \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 1 & -7 & 2 \end{bmatrix} \qquad 2.\ \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 1 & 3 & 4 \end{bmatrix} \qquad 3.\ \begin{bmatrix} 1 & 2 & 0 & 5 \\ 3 & 4 & 1 & 7 \\ -2 & 5 & 2 & 0 \\ 0 & 1 & 2 & -7 \end{bmatrix}$$

Cramer's Rule

Let $AX = B$ be a linear system of n equations in n unknowns. If $|A| \neq 0$ then if $X = [x_1\ x_2\ x_3\ \cdots\ x_n]^T$ then

$$\boxed{x_i = \frac{|A_i|}{|A|} \qquad i = 1, 2, \ldots, n,}$$

where A_i is obtained by replacing the i^{th} column of A by B.

Exercise: Solve using Cramer's Rule:

1.
$$\begin{aligned} 2x + 4y + 6z &= 2 \\ x + 2z &= 0 \\ 2x + 3y - z &= -5 \end{aligned}$$

2.
$$\begin{aligned} 2x + 3y + 7z &= 2 \\ -2x - 4z &= 0 \\ x + 2y + 4z &= 0 \end{aligned}$$

3.
$$\begin{aligned} x + y + z - 2w &= 2 \\ 2y + z + 3w &= 4 \\ 2x + y - z + 2w &= 5 \\ x + y + w &= 4 \end{aligned}$$