Math 100 Lecture: Definite Integrals

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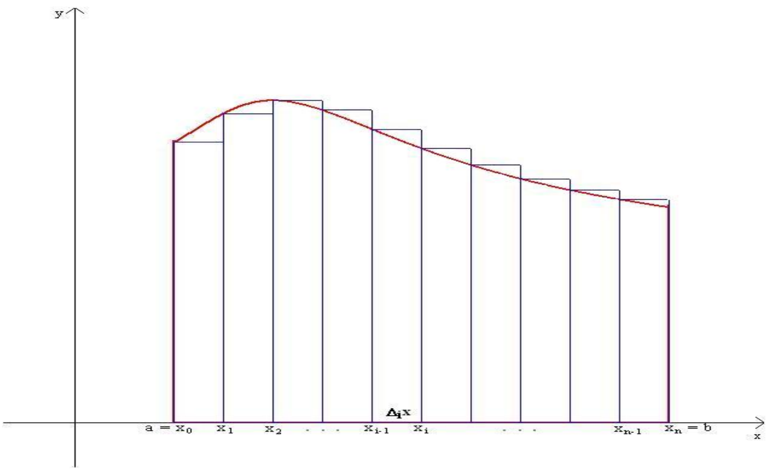
Lecture 27

ake an abstraction of what we have done previously. Let R be a region bounded by the function $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, $a < b$. Let Δ be a partition derived by subdividing the interval $[a, b]$ into n subintervals, not necessarily of equal sizes, such that $x_i = x_{i-1} + \Delta_i x$ for $i = 1, \dots, n$. Let $||\Delta||$ be the norm of the partition, that is, the largest width among the subintervals formed. Without loss of generality,we note that the height of each rectangle is determined by



$f(x_{i-1}), i = 1, \dots, n$. The area of each rectangle is given by

$$\Delta A_i = \text{ length } \times \text{ width } = f(x_{i-1})\Delta_i x \qquad i = 1, 2, \dots, n.$$

Summing up the areas, we get what is called the *Reimann sum*

$$\sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} f(x_{i-1})\Delta_i x.$$

As previously mentioned, the true area of the region is achieved by letting the $n \to \infty$, or equivalently, letting $||\Delta|| \to 0$:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta A_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta_i x = \lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_{i-1})\Delta_i x.$$

The rightmost side of the equality is called the limit of the Reimann sum and is more denoted by the simple $\int$.

$$A = \lim_{||\Delta||\to 0} \sum_{i=1}^{n} f(x_{i-1})\Delta_i x = \int_a^b f(x)\ dx$$

to be read as "the definite integral of f with respect to x from a to b."

Definition 1 *Let f be a function whose domain includes the closed interval $[a, b]$. Then f is said to be* <u>*integrable*</u> *on $[a, b]$ if there is a number L such that for any w_i in a subinterval $[x_{i-1}, x_i]$, $x_0 = a, x_n = b$, and for every partition Δ for which its norm is sufficiently small, then*

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x = L.$$

Definition 2 *If f is a function defined on the closed interval $[a, b]$, then the* <u>*definite integral*</u> *of f from a to b, denoted by $\int_a^b f(x)\ dx$, is given by*

$$\int_a^b f(x)\ dx = \lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(w_i)\Delta_i x$$

provided the limit exists.

Theorem 1 *If a function is continuous on the closed interval $[a, b]$, then it is integrable on $[a, b]$.*

Theorem 2 *We have the following properties for definite integrals:*

1. *If $a > b$, then $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$.*
2. *If $f(a)$ exists, then $\int_a^a f(x)\ dx = 0$.*
3. *If f is integrable on $[a, b]$ and if k is any constant, then $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$.*
4. *If f and g are integrable on $[a, b]$, then $f \pm g$ is integrable on $[a, b]$ and $\int_a^b [f(x) \pm g(x)]\ dx = \int_a^b f(x)\ dx \pm \int_a^b g(x)\ dx$.*
5. *If f is integrable on a closed interval containing three numbers a, b, and c, then*

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

regardless of the order of the numbers.

We now give the two fundamental theorems of integral calculus.

Theorem 3 (First Fundamental Theorem of Calculus) *Let f be a continuous function on $[a, b]$ and let x be any number in $[a, b]$. If F is a function defined by*

$$F(x) = \int_a^x f(t)\ dt$$

then

$$F'(x) = f(x) \Longleftrightarrow \frac{d}{dx}\int_a^x f(t)\ dt = f(x).$$

Theorem 4 (Second Fundamental Theorem of Calculus) *Let f be a continuous function on $[a, b]$ and let F be a function such that*

$$F'(x) = f(x) \qquad \forall x \in [a, b].$$

Then

$$\int_a^b f(t)\ dt = F(b) - F(a).$$

Exercises:

1. Perform the following:

   (a) $\displaystyle\int \left(\sqrt[4]{x+1} - \frac{x^2+1}{\sqrt{x}} \right) \, dx$

   (b) $\displaystyle\int \sec^3 x \;\sin 2x \; dx$

   (c) $\displaystyle\int \frac{x^2}{\sqrt{1-x}} \, dx$

   (d) $\displaystyle\int \frac{1-2\sqrt{x}}{\sqrt{x}(3+2\sqrt{x})^3} \, dx$

2. Evaluate the following definite integrals:

   (a) $\displaystyle\int_0^1 (3x^2 + 2\sqrt{x} + 3\sqrt[3]{x}) \, dx$

   (b) $\displaystyle\int_{-1}^0 (x+1)^3 \, dx$

   (c) $\displaystyle\int_{-1}^2 (3x^2 + 2x + 4) \, dx$

   (d) $\displaystyle\int_1^4 \frac{x^2-1}{\sqrt{x}} \, dx$

   (e) $\displaystyle\int_0^{\pi} \sin^2 x \cos x \, dx$

   (f) * $\displaystyle\int_{-2}^2 |1-x| \, dx$

   (g) * $\displaystyle\int_{-3}^3 |3x-2| \, dx$

   (h) * $\displaystyle\int_0^6 \sqrt{6x-x^2} \, dx$